THE SOCIETY ESTABLISHES A NEW PERIODICAL

Through the munificence of two of the great Foundations, the American Mathematical Society is in the fortunate position of having the financial backing to found a new international mathematical abstracting journal to be known as *Mathematical Reviews*. During the past quarter-century, while the United States and Canada have been gradually assuming a more prominent part in mathematical research, there has been sentiment expressed from time to time among mathematicians that there should be an abstract journal sponsored by American organizations. But the doubts whether we had the scientific and financial resources to spare caused the postponement of the undertaking. However, the rapid growth of our mathematical resources and the availability of funds have resolved these doubts and it has been decided to proceed immediately.

At the Annual Meeting of 1938 the Council of the Society appointed a committee consisting of C. R. Adams (chairman), G. D. Birkhoff, A. B. Coble, Thornton C. Fry, Marston Morse, and G. T. Whyburn with power to proceed with the establishment of a new journal provided the finances could be assured for a period of five years and provided international coöperation could be obtained. After much correspondence and several meetings and after consulting representative mathematicians in all parts of the country, the Committee decided on May 30, 1939 to proceed; and this decision was later ratified by the Council and by the Board of Trustees. An executive subcommittee consisting of Oswald Veblen (chairman), Thornton C. Fry, and Warren Weaver was appointed to officiate in setting up the machinery to get the journal underway, and this group will function until the Society makes permanent arrangements for the direction of the journal. The Council at the September meeting took initial steps to change the by-laws of the Society so as to define the relationship and responsibility of the Society to the new journal.

It is confidently expected that the assumption of this important task will stimulate research and teaching. Coöperation from the great fraternity of mathematicians is assured. Several hundred people have signed petitions requesting that the journal be inaugurated. Meetings of groups in various parts of the country have been held to discuss the desirability of the project and great interest has been evinced. Prominent mathematicians both here and abroad have given assurances that collaborators will be available in abundant numbers to carry on the journal at a high level.

The Carnegie Corporation has appropriated $60,000 as a backlog for the new journal. The Rockefeller Foundation has made a gift of $12,000 to cover some of the initial costs. Brown University is housing the project and aiding in the editorial work. The American Mathematical Society and the Mathematical Association of America are each starting off with a subsidy of $1,000 for the first year. Annual subsidies are being sought from other organizations, with prospects of success. Plans for the permanent financing of the project are also being considered.

The first number of *Mathematical Reviews* is to appear late in 1939 or early in 1940; the material to be reviewed begins with the latter half of 1939. It is proposed to review all fields of pure mathematics and also those parts of applied mathematics and mathematical physics which are of pronounced interest to mathematicians. The new journal, which will be issued approximately once a month, will contain several thousand reviews annually and will run to approximately eight hundred large double-column pages, each page being equivalent to nearly two of this *Bulletin*. Professors J. D. Tamarkin and O. Neugebauer will be the first editors. A strong group of collaborators for the initial period is assured. No salaries will be paid to the editors.

The languages to be used in the new journal are English, French, German, and Italian. It is planned to print a large edition of the first number and to circulate it among mathematicians throughout the world with a blank form calling for subscriptions. On account of the generous subventions, the price will be set drastically below actual cost. It has been decided to set the price at $13.00 with reductions to $6.50 for members of the supporting organizations and $3.25 to reviewers.

Partly with a view to aiding indirectly in the support of this journal, the Rockefeller Foundation has made a handsome gift to Brown University for an experiment in the dissemination of mathematical publications through the distribution of microfilm. This money is to be used to augment the mathematical library at that University, a collection which is already internationally known as outstanding. Out-of-print journals will be put on film and made available to mathematicians; rare books of general use will be filmed; on request from a subscriber to the new journal, any article reviewed will be sent on film or as film-print. This service will be extended to all parts of the world at a price not exceeding cost. It should be of greatest value to mathematicians located in the smaller universities and colleges and should be a factor in encouraging young men and women

to continue with their investigations. This interesting experiment in the promotion of a new aid to learning should prove to be an asset not only to *Mathematical Reviews*, but also to American mathematics in general.

Authors are requested to send reprints or preprints of all their research articles, beginning July 1, 1939, and as soon as they are available, to *Mathematical Reviews*, American Mathematical Society, Brown University, Providence, R. I. This will aid in getting reviews made promptly. Subscriptions may be sent to the New York office of the Society.

R. G. D. RICHARDSON,
Secretary

THE COLLECTED WORKS OF MILLER

The Collected Works of George Abram Miller. Vol. 2. By H. R. Brahana, R. D. Carmichael and A. B. Coble, The Committee on Publication, University of Illinois Press, 1938. 11+537 pp.

This volume reproduces one hundred seven papers published by Miller during the years from 1900 through 1907. It also contains two papers written by him especially for this volume. One of these (No. 63) is entitled "Note on the History of Group Theory during the Period Covered by this Volume"; the other (No. 171) is entitled "Primary Facts on the History of Mathematics." The second of these was included at Miller's request. In an appendix there are listed the titles of forty other papers written by him during the period; some of these are covered by papers included and the others are concerned with history, elementary illustrations, applications, or expositions. A subject index and a name index prepared by Miller are also included.

The paper entitled "Note on the History of Group Theory during the Period Covered by this Volume," like the three historical notes written for Volume I, gives a good survey of those developments which are closely related to Miller's work. In 1898 Miller published his first report on recent progress in the theory of groups of a finite order. Second and third reports on the same subject were published by him in 1902 and 1907 respectively. The second report (No. 89 in this volume) is devoted chiefly to recent advances in abstract groups, holomorphisms, substitution groups, and group characteristics, and the third (No. 153) is devoted to elementary applications, abstract groups, groups of isomorphisms, and substitution groups. These reports along with the historical papers just mentioned trace in a satisfactory way the development of the theory of groups up to 1908 and afford a splendid basis for the appreciation of the work done in this field up to this time by Professor Miller. Another paper (No. 64) is entitled "Report on Groups of an Infinite Order." In this paper considerable attention is given to a memoir on groups of movements written by Jordan in 1868 and brief accounts are given of the work done by Lie, Klein, Dyck and others; this paper supplements the reports on groups of a finite order.

To present the nature of the work contained in the 104 papers reproduced in this volume and not mentioned above, it will be convenient to consider them under the headings: (1) Papers on the determination of groups of finite order, (2) Papers which deal with the theory of groups, (3) Papers on group theory and number theory, (4), Reviews; elementary and expository papers; applications. These will be considered in order.

(1) *Papers on the Determination of Groups of Finite Order.*

These papers will be considered under the subheadings (a) Groups of given degree or order, (b) Groups containing given subgroups, (c) Groups having given defining relations, (d) Groups having given groups of isomorphisms.

(a) *Groups of given degree or order.* The papers in Volume I on the determination of groups of finite order are concerned chiefly with permutation groups of given degree and abstract groups of given order. In Volume II there are only five such papers. In one of these (written with G. H. Ling) a complete determination is made of the intransitive substitution groups of degree eleven. In another, all the abstract

groups of order 168 are found. The other three are devoted to simple groups; in one of these (written with G. H. Ling) it is proved that there is no simple group whose order lies between 1902 and 2001, in another it is shown that the degree of a simple group of odd composite order must exceed fifty, and in a third the groups of isomorphisms of the simple groups whose degrees are less than fifteen are determined.

(b) *Groups containing given subgroups.* In the study of permutation groups, especially of those which are primitive, the subgroups which leave one element fixed have played a noteworthy part. In this volume several important papers are devoted to the study of abstract groups with given subgroups or given operators. Five of these deal with the determination of all groups of order p^m (p a prime) which contain the Abelian group of type $(m-2, 1)$, the Abelian group of order p^{m-1} and type $(1, 1, \cdots, 1)$, operators of order p^{m-2}, exactly p cyclic subgroups of order p^{m}, or exactly $p+1$ Abelian subgroups of order p^{m-1}. Two papers deal with groups of order 2^m which contain an invariant cyclic subgroup of order 2^{m-2}, or an odd number of cyclic subgroups of composite order. An important set of papers deals with groups in which every subgroup is Abelian (this with H. C. Moreno), every subgroup of composite order is invariant, every subgroup is Abelian or Hamiltonian, every subgroup is Abelian or dihedral, or which contain less than six cyclic subgroups of the same order. Three other important papers are devoted to a generalization of the Hamiltonian groups, the invariant subgroups of prime index, and groups containing only three operators which are squares. Three papers relate to groups containing the largest possible number of operators of order two, thirteen operators of order two, or less than fifteen operators of order two; one paper to the groups which contain less than twenty operators of order three; and one paper to the possible number of operators of order two in a group of order 2^m. The remaining papers under this heading relate to the construction of a group of order p^q which includes no Abelian subgroup of order p^4, and to showing that each member of the Mathieu system of triply transitive groups is uniquely determined by its subgroup which leaves one element fixed.

(c) *Groups having given defining relations.* Much attention has been given by investigators in the field of group theory to the study of the properties of groups whose defining equations are given and the present volume contains a number of papers on this topic. One of the most important of these deals with the generalization of groups of genus zero. The generalization is obtained by replacing two of the three ordinary relations which define such a group by a single relation while the third is kept unchanged. Another paper closely allied to groups of genus zero is one on the groups which are generated by two operations of orders two and four respectively whose commutator is of order two.

Another paper of fundamental importance deals with groups defined by the orders l, m of two generators and the order n of their product. In this paper it is shown that there is always an infinite number of corresponding groups except when two of these numbers are equal to two, or one is two, the other three, while the third is three, four, or five. Papers relating to special cases deal with groups generated by two operators of order three whose product is also of order three, with groups generated by two operators of orders two and three respectively whose product is of order six, with groups generated by two operators of order three whose product is of order four, and with groups generated by two operators. Another paper deals with two infinite systems of groups generated by two operators of order four.

Another set of papers in this category is devoted to the groups generated by two operators which have a common square, groups generated by operators which trans-

form each other into their powers, groups generated by two operators which transform each other into the same power, and groups generated by two operators such that each is transformed into its inverse by the square of the other. Another paper is devoted to groups generated by three operators each of which is the product of the other two.

(d) *Groups having given groups of isomorphisms.* In a paper in the Transactions devoted to this topic all possible groups are determined whose group of isomorphisms is either the symmetric group of order six, or the symmetric group of order twenty-four. The infinite system of groups is also determined whose group of cogredient isomorphisms is the symmetric group of order six.

(2) *Papers Which Deal with the Theory of Groups.*

These papers may be classified into those relating to permutation groups and those relating to abstract groups. Among the former that on the product of two substitutions is of fundamental importance by reason of the theorem: If l, m, n, are any three integers greater than unity, of which we call the greatest k, it is always possible to find three substitutions (L, M, N) of $k+2$ or some smaller number of elements and of orders l, m, n respectively such that $LM=N$. This theorem is the basis of several results on the determination of groups. In a second paper there is proved a fundamental theorem relating to the degrees of the transitive constituents of the Sylow subgroups of any transitive group. In another interesting paper it is shown that the group of the five times transitive function of 24 elements of Mathieu and its maximal subgroups of degrees 21, 22, and 23 are simple groups. Other papers on permutation groups relate to the transitive groups which are isomorphic to a given group, transitive groups of order p^m, primitive groups of class four, and the invariant substitutions under a substitution group.

The papers relating to the theory of abstract groups cover a wide range of topics. In one, on the groups which are the direct product of two subgroups, four important theorems are proved. In two others several new properties of the holomorph of a cyclic group are presented. Four papers relate to groups of isomorphisms; one to the group of isomorphisms of a cyclic group, one to that of a group of order p^m, and two to groups of isomorphism of Abelian groups. Three papers are on conformal groups; in one of these necessary and sufficient conditions are developed that any Abelian group be conformal with at least one non-Abelian group, in one an infinite system of non-Abelian groups which are not conformal with any Abelian group is studied, and in a third groups of order p^m which are not conformal with Abelian groups are considered.

Several papers on abstract group theory relate to subgroups. Important among these is one in which all the characteristic subgroups of any Abelian group are determined, and another in which a formula is developed for writing all the subgroups of such a group. In one of two papers concerning groups of order p^m it is proved that every non-Abelian group of this order contains at least p invariant commutator operators, and its commutator quotient group is always non-cyclic (this being an extension of a theorem by Sylow); in the other it is shown that every non-cyclic group of order p^m contains at least one invariant non-cyclic subgroup of order p^β, where $\beta>1$. In one of two papers concerned with the number of operators whose orders exceed two a formula is developed for this number in a non-Abelian group of order 2^m in which the number of operators of order 2 is more than half the order of the groups; in the other the minimum number of operators whose orders exceed two in any group of finite order is considered. Two papers relate to subgroups of order p

in groups of any given order; in one of them it is proved that if the order of a Sylow subgroup is p^m and if this subgroup is non-cyclic then the number of cyclic subgroups of order p^α is kp, and in the other an upper limit is found for the number of sets of conjugate subgroups of order p^α, where $\alpha < m$. Other papers concerning subgroups or operators are devoted to the invariant subgroups of index p^3, the number of Abelian subgroups whose order is a power of a prime, an addition to a theorem due to Frobenius, groups containing several operators of order two, and theorems relating to quotient groups.

Further papers on the theory of abstract groups are devoted to a method for constructing all groups of order p^m, to the roots of the operators of a group, to groups in which all the operators are contained in a series of subgroups such that any two have only the identity in common, to the commutators of a group of order p^m and to the definition of a complete group. Besides these a number of theorems of importance are proved in the papers noted above on the determination of groups of finite order.

(3) *Papers on Group Theory and Number Theory.*

Six papers in this volume are devoted to the study of properties relating to both group theory and number theory. In a paper on holomorphisms and primitive roots important properties of the group of isomorphisms of a cyclic group of order p^m are developed and used to prove the existence of primitive roots of p^m and $2p^m$ and also to show how to determine these roots. In another paper, theorems relating to the product of all the operators in an Abelian group and to the order of a cyclic group whose group of isomorphisms contains only one operator of order two provide the basis for a new proof of the generalized Wilson's Theorem. A third paper on some relations between number theory and group theory has close contact with the first paper above and others in the literature. Its chief object is to present certain additional developments which show the usefulness of group theory in the study of number theory. In two other papers elementary theorems in group theory are used to prove the formula for the totient of order k with respect to m, and other well known relations connecting totients. The remaining paper in this category aims to translate several theorems in number theory into the language of group theory rather than to exhibit the advantages of employing theorems in group theory in the proof of important results in number theory as is done largely in the other five papers.

(4) *Reviews; Expository and Elementary Papers; Applications.*

Two papers are devoted to reviews of treatises on group theory. In one of these a rather complete review of Burnside's *Theory of Groups* (first edition) is given; in the other a brief review of Dickson's *Linear Groups with an Exposition of the Galois Field Theory* is made.

Expository and elementary papers relate to some methods of constructing substitution groups, the product of two commutative operators, the definition of an infinite number, the generalization and extension of Sylow's theorem, what is group theory, the commutator of two operators, a note on Netto's *Theory of Substitutions*, groups of the fundamental operations of arithmetic, and groups of subtraction and division.

Applications are given in papers on a new chapter in trigonometry, and the use of group theory in elementary trigonometry.

The last paper (No. 171) is entitled "Primary Facts in the History of Mathematics." It is a new paper written for this volume and, as stated at the beginning of

this review, is included at Professor Miller's request. It presents some of the facts which guided him in his writings on the history of mathematics and affords a suitable basis for evaluating this phase of his work. The facts presented are discussed under the headings: abstractions, generalizations, wider outlook, special language, Greek contributions, thought currency, subjects treated, different treatments, fashions, independent discussions, formulas, and dependent subjects. This important paper will be of value and interest to students of the history of mathematics.

H. W. KUHN

SCHOUTEN AND STRUIK ON DIFFERENTIAL GEOMETRY

Einführung in die neuren Methoden der Differentialgeometrie. By J. A. Schouten and D. J. Struik. 2d completely revised edition. Vol. 1. *Algebra und Übertragungslehre.* By J. A. Schouten. 12+202 pp. Vol. 2. *Geometrie.* By D. J. Struik. 12+338 pp. Groningen, Noordhoff, 1935 and 1938.

As a revision and extension of the authors' previous treatises, this second edition of their joint work should be compared, not merely with the first edition, a short monograph published in 1924, but also with Struik's *Grundzüge der mehrdimensionalen Differentialgeometrie in direkter Darstellung*, published in 1922 and now out-of-print. The monograph dealt primarily with the algebraic and analytic aspects of the subject, whereas Struik's book was concerned essentially with Riemannian geometry.

Perhaps the most striking difference between the present work and those cited is in the methods employed. Schouten's system of direct analysis was used concurrently with the method of tensors in the original monograph and was employed almost exclusively in Struik's *Grundzüge*. In the book under review, this system has been completely discarded and tensor analysis given full sway.

In the notation for the components of geometrical objects Schouten's *Kern-Index-Methode* is consequentially applied throughout the treatise. The essence of this method consists in representing the effect on the components of a geometrical object due to a change of the system of reference, not by a different or primed central letter with indices of the original type, as is frequently the custom, but by the original central letter with indices of a different type. The change in the central letter with the preservation of the type of indices is reserved to represent the transformation of the object resulting from an actual transformation of the space. The method seems logically sound in principle and makes for definiteness and conciseness in practice. Readers to whom it is new will find that it is readily followed once they accustom themselves to scanning each formula for the full import of all the letters, central and appended, that are involved.

The outstanding advance in general methods since the appearance of the authors' first books has undoubtedly been the exploitation of nonholonomic systems of reference. These are introduced at the earliest opportunity in the first volume of the new work and are used throughout wherever fitting. In fact, the ordinary sign of equality is employed to indicate the validity of a relationship for all types of systems of reference. To indicate that an equation is true only for holonomic systems, an h is placed above the sign of equality, and a * serves a similar purpose in the case of an equation which holds only for the specific system of reference in use at the moment. Here, also, the ends of clarity and conciseness are well served.

Whereas the volumes of the early twenties treated only ordinary Riemannian geometry, the present work covers also the general linear connection and Hermitian connections, as well as the extension of Riemannian geometry to the general case in which the metric is not necessarily definite. In particular, in the latter connection, isotropic subspaces are given more than the customary passing mention.

The first volume begins with the algebraic foundation, including the fundamentals of affine geometry and of unitary geometry. The second part of this volume, in which the tools from analysis are developed, brings in at once nonholonomic systems of reference, develops the general linear connection with respect to them, discusses the D-symbolism of van der Waerden and Bortolotti in relation to the machinery for

treating subspaces, treats geodesics and normal coordinates, curvature tensors and related topics, and ends with a discussion of the differential operators of importance in problems of variation and deformation.

The second volume falls into four parts. The first, on curves, deals with individual curves with respect to both a Riemannian connection and the general linear connection, and considers congruences of curves and natural systems of curves in a Riemannian space V_n. The second treats hypersurfaces in a V_n, including the study of individual curves and congruences of curves which lie in them, and the Gauss-Codazzi equations. In the third part is the corresponding material in the case of a V_m in a V_n, together with a discussion of the conditions under which a V_m may be imbedded in a Riemannian space of constant curvature S_n, and applications of the Gauss-Codazzi-Ricci equations to special problems pertaining to the class of a V_m and the possibility of deforming it in an S_n. The last part treats a diversity of selected topics: the curvature properties of nonholonomic spaces with respect to the general linear connection; the fundamentals of the theory of infinitesimal deformation; isotropic subspaces, in general nonholonomic, of a V_n; path-preserving transformations of affine connections with an introduction to projective connections; conformal transformations and connections; Hermitian connections.

Scattered through each volume there are numerous problems, ranging from simple exercises designed for beginners to substantial problems incorporating the results of research. The solutions of the problems which are given at the ends of the volumes will be welcome both to novice and expert. Both volumes are provided with excellent bibliographies and indices.

The exposition is generally maintained on so high a plane that criticism, even of a minor nature, seems out of order. It should, nevertheless, be pointed out that the reader, and particularly the beginner, would have been better served if in the early pages of the first volume the material had not been so condensed, from the original edition, as to force important definitions into the footnotes and to eliminate entirely the discussion of the geometrical significance of contragredient transformations.

The term *second edition* must here be interpreted as reincarnation, both in flesh and in spirit. The new book should prove an invaluable addition to the literature of differential geometry and an incentive to further progress.

W. C. GRAUSTEIN

SHORTER NOTICES

Valeur Pratique et Philosophie des Probabilités. By É. Borel. (Traité du Calcul des Probabilités et de ses Applications, vol. 4, no. 3.) Paris, Gauthier-Villars, 1939. 9+182 pp.

The treatise on probability and its applications, published by É. Borel and distinguished collaborators has been brought to a close by the present book, in which Borel discusses the place of the calculus of probability in man's analysis of the world about him. Borel does not attempt a systematic formal presentation: the book is an agreeably written non-technical discussion of various crucial aspects of the subject. In Borel's opinion, all our knowledge is probability knowledge, and his whole discussion has that point of view. Thus he even considers the problem of evaluating the probability that the sun will rise tomorrow, or the probability *a priori* that astrology is correct. It is difficult sometimes to determine whether probability, as Borel uses the term, refers to an individual's feelings, as measured by odds he might be willing to accept on a certain event, or whether probability refers to an event occurring in accordance with given specified conditions. Thus Borel considers a "reasonable person" R who is told that 12 horses are to race, who knows nothing about the horses, but who is nevertheless asked the probability that a certain horse A will win. According to Borel, R would try to avoid making any judgment in such a vaguely defined situation, but if pressed, could only give the answer 1/12. It seems to the reviewer that if R knows nothing about the horses, he also knows nothing about the probabilities concerned, and that he must necessarily be silent on the whole situation. If the answer 1/12 is proper why should R be reluctant to give it? The answer 1/12 is not really an answer to the question. In fact the individual who answers 1/12 has constructed a new problem: given that an event can occur in 12 ways, the conditions of occurrence being such that the theory of probability is applicable, and such that no one manner of occurrence is favored over another, what is the probability that the event will occur in way A? The answer is of course 1/12; in fact the question was framed with that answer in mind. But the question is not the original one, because the relevant features of the frame of reference of the experiment have been made specific in the second version, while no useful hint of the real frame of reference was given in the first version.

At one point Borel makes a more or less formal definition. Probability is defined subjectively for an individual by the conditions of a bet he would accept. An objective probability value, to Borel, is a value which is the same "for a certain number of individuals equally well-informed on the conditions of the chance event." Since these betting odds are judged by the individuals concerned using their past experience and their desire to profit in the long run, their probability values can be checked by comparison with observed frequencies, which may suggest changed estimates, or even changes in the technique of estimating.

Estimating probabilities is thus the expected combination of experiment and rational thinking. There seems little point in arguing over definitions of probability as an empirical concept. The exact way probability is defined is of little importance as against the way the numbers when obtained are checked and used. The reviewer can see no possible use or meaning in a number which is called the probability that the sun will rise tomorrow. If one adopts the view that an event to which the probability calculus is applicable should have certain characteristics, leading to an actual

estimate of the probability concerned, then the probability calculus is certainly inapplicable here. If one speaks of the sun's rise as an event which is uncertain (like everything else) and therefore as an event which has a probability number, just what physical meaning has such a number? A short review, however, is no place to take issue with Borel's subjectivism; even a reader who disagrees with much of the material can not deny its thought provoking character, and such a reader will very probably enthusiastically endorse some interesting remarks by Darmois in an appended note.

J. L. DOOB

Partielle Differentialgleichungen und ihre Anwendungen auf physikalische Fragen. By B. Riemann. (Edited by K. Hattendorff, with introduction by F. Emde.) Braunschweig, Vieweg, 1938. 12+325 pp.

It is a tribute to Georg Friedrich Behrnard Riemann to have his book written in 1882, itself only a slight revision of the first edition of 1869, reprinted unaltered in 1938. The following explanation is given by Fritz Emde:

"In the course of time a very detailed two volume work on which many authors have cooperated has grown out of Riemann's lectures on partial differential equations (for review of the eighth edition see this Bulletin, vol. 37 (1931), p. 333). The original edition looks quite modest in comparison with this revision. In spite of this engineers and physicists have repeatedly asked for this book which has been out of print for some time, and were justified in doing so since it is a book in which Riemann introduces his readers to the fundamental mathematical ideas in an excellent way and teaches them the methods of solution. For the beginner even today there is hardly a more convenient approach to this subject.

"May Riemann's lectures show their old virtues anew. They will be a credit to any collection of books."

The reviewer cannot, however, refrain from warning against some concepts, for example (see page 9) dx is considered an "infinitely small" quantity.

J. F. RANDOLPH

Geometrie der Gewebe. By W. Blaschke and G. Bol. Berlin, Springer, 1938. 8+339 pp.

During the years 1927 to 1938 there appeared, mostly in the Hamburg Abhandlungen, a series of papers under the general title, "Topologische Fragen der Differentialgeometrie." The authors of "Geometrie der Gewebe" have been the most frequent contributors to this series, and in this book they have systematized and amplified the theory that was built up in these papers.

The basic concept in web geometry is the *sheaf of curves*, a topological image of the portion of a pencil of parallel lines contained in a bounded convex region of the plane of the pencil. An *n-web* is a set of n sheaves of curves, the points on each sheaf constituting the same point set G, such that no two curves of different sheaves have more than one common point. The property of being an n-web is evidently preserved under any topological transformation of G, and the theory of webs concerns itself largely with properties of webs which are invariant under such transformations. The central problem is this: Under what conditions can an n-web be mapped topologically into a set of line segments in a plane, and how can these webs of lines be characterized?

These considerations can easily be extended to configurations of higher dimensionality. Thus in three dimensions we may have a sheaf of surfaces which is the topological image of the intersection of a pencil of parallel planes with a convex region

of 3-space, or a sheaf of curves obtained similarly from a bundle of parallel lines. Other higher dimensional analogues are evident, but these two seem to be the only ones whose properties have been studied to any great extent.

The first part of the book is devoted mainly to "hexagonal" webs, in which there exist certain configurations formed by the curves of the web (analogous to the configurations of Desargues or Pappus in projective geometry). The principal result is the following:

Any hexagonal n-web can be mapped onto n pencils of lines in a plane if $n \neq 5$. There exist 5-webs which cannot be so mapped, but these can be mapped onto four pencils of lines and the pencil of conics on their centers.

This theorem is proved for $n=3$ (the first non-trivial case) by topological methods, and then extended to higher values of n by the use of properties of continuous groups. Similar theorems are obtained for the two 3-dimensional webs mentioned above.

In the remaining two parts of the book the discussion is limited to those webs and transformations for which the defining functions have a suitable number of derivatives. Part II treats of the differential invariants of webs; including among other topics the determination of complete sets of invariants, a characterization of hexagonal webs, and the relation between webs and the theory of parallelism in differential geometry. Extensive use is made here of the properties of Lie groups and the differential operators arising from them. In Part III webs are discussed from the point of view of algebraic geometry, the main tools being Abel's theorem and the properties of Abelian integrals. The *rank* of a web is introduced as the number of ways of setting up certain parametrizations of the curves of the web. The principal theorems in this connection are the following:

If an n-web of lines in a plane has positive rank, it consists of tangents to an algebraic curve (possibly reducible) of class n.

Any 3- or 4-web of maximal rank can be mapped on a web of lines in a plane.

Probably the most striking feature of the book is the way in which so many different branches of geometry are applied to the development of the theory of webs. There are numerous applications of theorems in topology, continuous groups, algebraic geometry, and differential geometry. These theorems are not merely referred to, but proofs of them are given, at least for the special cases which are to be used. As a result the book is almost wholly self-contained, in spite of the wide range of the topics that are touched upon. This feature, and the inclusion of numerous exercises which supplement and extend the expounded material, aid in making the book an excellent introduction to an interesting new chapter in geometry.

R. J. WALKER

Analyse Mathématique. Vol. 2. *Équations Différentielles. Développements en Séries. Nombres Complexes. Intégrales Multiples.* By P. Appell. 5th edition completely revised. (Cours de Mathématiques Générales.) Paris, Gauthier-Villars, 1938. 305 pp.

The author devotes this second volume of his set on Mathematical Analysis to the following topics in the order mentioned: Differential Equations (91 pages, 3 chapters); Developments in Series (80 pages, 2 chapters); Complex Numbers (39 pages, 1 chapter); Multiple Integrals (78 pages, 2 chapters). The written text and the two appended notes are supplemented with 79 well-drawn figures and are divided into 177 sections each with a title and number. A table of contents, but no index, is bound with this second volume. No lists of exercises are included. However the various topics are illustrated by examples worked out by the author and these examples

are liberally scattered throughout the volume. In fact not fewer than 14 of the numbered sections have the word "exemples" or "exemple" as a part or all of their titles.

The usual material on first order differential equations is given ending with a 10 page brief discussion of the general theory. To treat second order equations the author considers first those that are reducible to first order equations, next linear equations, next differential systems of the second order and then gives a few remarks about systems of n first order equations.

After deriving and illustrating Taylor's formula, applications are made to calculating approximate function values and to finding limits of indeterminate forms and order of contact of two curves. Infinite series are introduced, and in turn, power series. Application is made to the calculation of function values, to integration taking the rectification of the ellipse as an example, and to solving differential equations.

Defining a complex number as a number pair (a,b) rules of operation are developed and graphically illustrated. De Moivre's formula follows and then the theory developed is applied to solving second and third degree polynomial equations. Series of complex terms are introduced, e^z, $\sin z$, and $\cos z$ are defined and Euler's formulas obtained. Some applications are made to second order differential equations.

The double integral is defined as a limit and this same limit is shown to be a volume. It is shown to equal the iterated integral and its calculation illustrated in polar and rectangular coordinates. Areas of curved surfaces, infinite domains of integration and application of double integrals to simple integrals containing a parameter are taken up. Riemann's formula connecting a double integral and curvilinear integral is then dealt with briefly. The triple integral is defined as a limit of a sum, analogously to the double integral, and is applied to potentials, to centers of gravity, moments of inertia and surface integrals. A 10 page section on elements of vectorial analysis, including mention of Green's theorem and the formulas of Stokes and Ampère, concludes the volume.

A 7 page appended note gives an interesting discussion of the theory of mechanical integrators. A second 4 page note is given to developing and illustrating L'Hospital's rule.

To the reviewer the volume seems to be carefully written and readable. Only a few minor, but obvious, misprints were noticed.

H. E. SPENCER

NOTES

The third annual mathematical symposium at the University of Notre Dame was held on April 12–13, 1939, on metric geometry. The following papers were presented: *The characterization of pseudo-spherical sets, with applications to determinant theory*, L. M. Blumenthal, University of Missouri; *Some recent results on pseudo-plane quintuples*, P. M. Pepper, University of Notre Dame; *A new mode of a locally euclidean surface*, M. Sadowsky, Armour Institute of Technology; *A new characterization of the straight line*, L. M. Blumenthal; *Metric spaces related to the theory of critical points*, Marston Morse, Institute for Advanced Study; *Three metrics arising in the calculus of variations*, Karl Menger, University of Notre Dame; *Some analytical aspects of isometric imbedding*, I. J. Schoenberg, Colby College; *Algebraic treatment of the isometric imbedding into euclidean and related spaces*, Emil Artin, Indiana University and the University of Notre Dame; *On the isometric imbedding into groups*, B. J. Topel, University of Notre Dame; *Universal ordered sets*, A. N. Milgram, University of Notre Dame; *A general theory of relation-preserving imbedding*, Karl Menger.

Professor A. Sommerfeld of the University at Munich has been awarded the Lorentz Medal of the Royal Dutch Academy of Sciences.

On the occasion of the celebration of the fiftieth anniversary of the founding of the University at Sofia, honorary doctorates were awarded to the following mathematicians: Dean G. D. Birkhoff, Harvard University; Professor Wilhelm Blaschke, University of Hamburg; Professor Émile Borel, Sorbonne; Professor Emeritus A. L. Bowley, University of London; Professor G. H. Hardy, University of Cambridge; Professor Paul Montel, Sorbonne; Professor Waclaw Sierpiński, University of Warsaw.

Professors E. Cartan, J. A. Schouten, D. van Dantzig, W. Fock, and B. Kagan have been elected honorary members of the Mathematical Society of Kazan.

Dr. A. P. Guinand of the University of Adelaide, Oxford, and Mr. D. G. Northcott of St. John's College, Cambridge, have been awarded Commonwealth Fund fellowships. They will study at Princeton University.

Professor H. Lebesgue of the College of France has been granted an honorary doctorate by the University at Lwow and by the University at Cracow. Professor Lebesgue has become a member of the editorial committee of Enseignement Mathématique.

Professor M. Picone of the University of Rome has been elected a member of the Royal Academy of Sciences.

Professor S. Timoshenko has been elected correspondant of the French Academy of Sciences.

Professor E. E. DeCou of the University of Oregon has been given the title emeritus. He has retired as head of the department of mathematics but will continue to do some teaching.

Harvard University has conferred an honorary doctorate upon Dr. P. W. Bridgman, professor of mathematics and natural philosophy at Harvard University.

Professor Joseph Bowden of Adelphi College, Garden City, New York, has been given the title emeritus. He will continue to lecture in mathematics.

Professor Otto Dunkel of Washington University has retired with the title emeritus.

Dr. O. G. Harrold has been appointed to a National Research Council fellowship. He will work at the University of Virginia.

The University of Missouri has conferred an honorary doctorate upon Dr. E. R. Hedrick, provost and vice president of the University of California at Los Angeles.

Mr. I. W. Smith has been given the title of professor emeritus by North Dakota State College in Fargo.

Professors Oswald Veblen and Hermann Weyl of the Institute for Advanced Study have been elected honorary members of the London Mathematical Society.

Professor A. Huber has been appointed to a professorship at the University of Vienna.

Dr. Max Müller of the University of Heidelberg has been appointed to an assistant professorship at the University of Tübingen.

Dr. Ernest Peschl of the University of Bonn has been promoted to an assistant professorship.

Professor A. Pflüger has been appointed to a professorship at the University of Freiburg.

Professor W. Schmeidler of the University of Breslau has been appointed to a professorship at the Technical School in Berlin.

Professor William Threlfall of the University of Halle has been appointed to a professorship at the University of Frankfurt.

Mr. P. Vincenzini has accepted a position at the University of Besançon.

Associate Professor V. W. Adkisson of the University of Arkansas has been promoted to a professorship.

Dr. C. B. Allendoerfer of Haverford College has been promoted to an assistant professorship.

Mr. H. F. Archibald of Keuka College, Keuka Park, New York, has been promoted to an assistant professorship.

During 1939–1940 Professor Emil Artin will give monthly lectures in group theory and theory of numbers at the University of Notre Dame.

Assistant Professor J. L. Barnes of Tufts College has been promoted to an associate professorship.

Dr. P. O. Bell of the University of Kansas has been promoted to an assistant professorship.

Dr. G. D. Birkhoff, Perkins professor of mathematics at Harvard University, retired on September 1, 1939, as dean of the faculty of arts and sciences.

Dr. W. Z. Birnbaum of New York University has been appointed to an assistant professorship at the University of Washington.

Assistant Professor L. M. Blumenthal of the University of Missouri has been promoted to an associate professorship.

Assistant Professor Salomon Bochner of Princeton University has been promoted to an associate professorship.

Assistant Professor H. F. Bohnenblust of Princeton University has been promoted to an associate professorship.

Dr. W. D. Cairns, professor of mathematics at Oberlin College for the past forty years, has retired. He will continue to serve the Mathematical Association of America as secretary-treasurer, with office at Oberlin, Ohio.

Assistant Professor Alonzo Church of Princeton University has been promoted to an associate professorship.

Mr. O. C. Collins of the University of Nebraska has been promoted to an assistant professorship.

Assistant Professor T. F. Cope of Queens College has been promoted to an associate professorship.

Dr. A. P. Cowgill of Syracuse University has been promoted to an assistant professorship of applied mathematics.

Assistant Professor D. C. Dearborn of Catawba College, Salisbury, North Carolina, has been promoted to an associate professorship.

Dr. K. S. Ghent of the University of Oregon has been promoted to an assistant professorship.

Associate Professor R. F. Graesser of the University of Arizona has been promoted to a professorship and to the chairmanship of the department of mathematics.

Dr. Israel Halperin of Harvard University has been appointed to an assistant professorship at Queen's University, Kingston, Ontario, Canada.

Dr. R. A. Higdon of Iowa State College has been promoted to an assistant professorship.

Dr. Grace M. Hopper of Vassar College has been promoted to an assistant professorship.

Dr. I. M. Hostetter of Howard College has been promoted to a professorship.

Assistant Professor R. D. James of the University of California has been appointed to a professorship and the chairmanship of the department of mathematics at the University of Saskatchewan.

Dr. Caroline A. Lester of the New York State College for Teachers has been promoted to an assistant professorship.

Dean G. H. Ling of the University of Saskatchewan has retired.

Assistant Professor M. H. Martin of the University of Maryland has been promoted to an associate professorship.

Associate Professor A. D. Michal of the California Institute of Technology has been promoted to a professorship.

Dr. Harriet F. Montague of the University of Buffalo has been promoted to an assistant professorship.

Associate Professor A. F. Moursund of the University of Oregon has been named chairman of the department of mathematics.

Assistant Professor Rufus Oldenburger of Armour Institute of Technology has been promoted to an associate professorship.

Dr. T. S. Peterson of Shipley School, Bryn Mawr, Pennsylvania, has been appointed to an assistant professorship at the University of Oregon.

Dr. Everett Pitcher of Lehigh University has been promoted to an assistant professorship.

Assistant Professor G. B. Price of the University of Kansas has been promoted to an associate professorship.

Associate Professor P. R. Rider of Washington University has been promoted to a professorship.

Dr. A. E. Ross of St. Louis University has been promoted to an assistant professorship.

Assistant Professor H. C. Shaub of Washington and Jefferson College has been promoted to an associate professorship.

Associate Professor H. M. Sheffer of Harvard University has been promoted to a professorship.

Dr. A. H. Smith of Purdue University has been promoted to an assistant professorship.

The following appointments to instructorships are announced: Adelphi College: Dr. H. J. Riblet; Alborz College (Teheran, Iran): Mr. E. S. Kennedy; Brown University: Dr. D. W. Hall, Mr. L. B. Hedge; Cornell University: Dr. Fritz Herzog; Duke University: Dr. R. P. Boas, Jr.; Emory University: Dr. G. B. Lang; Harvard University: Mr. M. H. Heins, Mr. E. N. Nilson; Haverford College: Mr. E. E. Betz; University of Minnesota: Mr. Fulton Koehler; New York University: Dr. H. E. Robbins; Oberlin College: Dr. R. W. Wagner; Ohio State University: Mr. M. Hendrickson, Dr. P. V. Reichelderfer, Dr. W. S. Snyder; University of Oregon: Mr. C. F. Kossack; Princeton University: Mr. J. W. Tukey; Purdue University: Dr. C. E. Clark, Mr. J. J. Eachus; Rensselaer Polytechnic Institute: Dr. B. A. Lengyel; Stanford University: Dr. A. C. Schaeffer; Wells College: Dr. Dorothy Manning; University of Wyoming: Mr. T. C. Doyle.

The death of Professor H. H. Barnum, head of the department of mathematics at Robert College, Istanbul, Turkey, is reported. He was sixty-one years of age.

The death of F. C. Boon, who was from 1909 to 1936 principal master in mathematics at Dulwich College, is reported in the Mathematical Gazette.

Professor Emeritus Philipp Furtwängler of the University of Vienna died April 21, 1939, at the age of seventy.

Professor Joseph Klobouček of the University of Prague died on April 18, 1939, at the age of sixty-four years.

Dr. S. L. Loney, who was professor of mathematics at the University of London from 1888 until 1920, died on May 16, 1939, at the age of seventy-nine years.

Professor E. Meissner of the Zurich Polytechnic School died March 18, 1939, at the age of fifty-six years.

Professor Pilo Predella died January 4, 1939, at the age of seventy-five years.

Professor H. Schotten, former editor of Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, died in Berlin at the age of eighty-two years.

Professor G. Tzitzéica of the University of Bucharest died February 4, 1939, at the age of sixty-five years.

Professor T. E. Mason of Purdue University died May 26, 1939. Professor Mason had been a member of the Society since 1913.

Dr. W. C. Risselman died at Inglewood, California, on June 26, 1939 at the age of thirty-nine years. He was head of the department of mathematics at Arizona State Teachers College, Flagstaff, from 1929 to 1937 when he was taken ill. His membership in the Society dated from 1928.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

290. Walter Leighton: *Proper continued fractions.*

This paper presents a generalization of the regular continued fraction which includes the "continued cotangent" algorithm of Lehmer as a special case. It is shown that corresponding to any sequence of positive integers $a_1, a_2, a_3, \cdots$ there exists a unique *proper* continued fraction expansion of the form $b_0+K(a_n/b_n)$ for each real number y_0. Here b_0 is an integer and $b_1, b_2, b_3, \cdots$ are positive integers with $b_n \geqq a_n$. The generalizations of a number of theorems pertaining to the regular case are obtained, including a generalization of the famous Lagrange theorem on the representation of a quadratic surd. (Received June 3, 1939.)

291. A. N. Lowan: *On the problem of wave-motion for the wedge of an angle.*

The object of this paper is the integration of the nonhomogeneous differential equation of wave-motion for the two-dimensional domain bounded by the half planes $\theta=0$ and $\theta=\theta_0$. In Part I the displacement is prescribed over both boundary planes. In Part II the displacement is prescribed over one of the boundary planes while the gradient of the displacement is prescribed over the other. Finally in Part III the gradient of the displacement is prescribed over both boundary planes. The solution of the systems satisfied by the Laplace transform are obtained by means of Green's theorem in terms of the appropriate Green's functions for the operator ∇^2+k^2. In Parts I and III the required Green's functions are obtained from the integral representations of arbitrary functions in terms of the Green's functions in the theory of heat conduction for the same domain. The Green's function in Part II is obtained from that of Part I by a reflection of the given domain on the line $\theta=\theta_0$. The transition from the Laplace transform to the functions themselves offers no difficulty. (Received June 16, 1939.)

292. A. N. Lowan and Jack Laderman: *On the distribution of errors in nth tabular differences.*

The object of this paper is the determination of the distribution of errors in the nth tabular differences of a mathematical table under the assumption of a rectangular distribution of the errors in the entries of the table, the entries being given to k decimal places. This problem is solved with the aid of the powerful theorem which states that the characteristic function of the distribution of a sum of random variables is equal to the product of the characteristic functions of the distributions of the individual

variables. The characteristic function of the desired distribution is readily obtained and the formal integral representation of the frequency function is obtained by the well known formula for the inversion of the Fourier transform. The frequency function is an even function, vanishing for $|x| \geq 2n^{-1} \cdot 10^{-k}$ and is defined by different polynomials in different intervals. Explicit expressions for the frequency function are given for the cases $n=1$, $n=2$, and $n=3$. The results for $n=2$ have been applied to an analysis of the errors in the second tabular differences of the function $\mathrm{Ci}(x)$ computed by the New York City WPA Project for the Computation of Mathematical Tables. (Received June 23, 1939.)

293. R. P. Agnew: *On kernels of faltung transformations.*

The main theorems proved are of the following type: If a kernel $J(t)$ is such that the faltung $y(s)=\int_{-\infty}^{\infty} J(t)x(s+t)dt$ has property P for each function x in class X, then $J(t)$ must have property Q. In each case, X is a certain class of nonnegative integrable generalized step-functions. If P is the property that $y(s)$ exists for at least one s, then Q is the property $\int_u^{u+A} |J(t)|\,dt$ is a bounded function of u for each $A>0$. If P is the property $y(s)$ is integrable, then Q is the property $J(t)$ is integrable. If P is the property $y(s)$ is essentially bounded, then Q is the property $J(t)$ is essentially bounded. Each of these theorems is supplemented by a theorem of familiar type which asserts that if J has property Q then (i) y has property P for each integrable function x and (ii) a certain constant determined by J is the *bound* of the transformation, that is, the least constant M such that a constant (norm) determined by y is less than or equal to $M\int_{-\infty}^{\infty} |x(t)|\,dt$ for each integrable function x. (Received June 26, 1939.)

294. R. P. Agnew: *On translations of functions and sets.*

If a sequence $x_1(t), x_2(t), \cdots$ of complex-valued functions measurable over $-\infty<t<\infty$ is such that, for each real sequence $\lambda_1, \lambda_2, \cdots$ (i) $\lim_{n\to\infty} x_n(t-\lambda_n)=0$ for each t in some set C of positive measure $|C|$, then, for each $\delta>0$, (ii) $\sum_{n=1}^{\infty}$ l.u.b. $|E_t\{h\leq t\leq h+1; |x_n(t)| \geq \delta\}| < \infty$ where l.u.b. is taken over $-\infty<h<\infty$. Conversely if $x_1(t), x_2(t), \cdots$ is a sequence of measurable functions for which (ii) holds, then for each sequence $\lambda_1, \lambda_2, \cdots$ (i) holds for almost all t in $-\infty<t<\infty$. Some lemmas on translations of sets, which are somewhat more incisive than those required for proof of the direct theorem, are proved and discussed. (Received July 31, 1939.)

295. Warren Ambrose: *Some properties of measurable stochastic processes.*

Some general conditions are found for the existence of what Doob has called a measurable stochastic process (Transactions of this Society, vol. 42 (1937), pp. 107–140). (Received August 2, 1939.)

296. H. A. Arnold: *Defective groups.*

In the usual postulates for a group, the closure postulate is altered so that it is required that $a \cdot b$ be in the group whenever a and b are in the group and are different. The mark $a \cdot a$ may or may not be an element for any a. The structure of the resulting system is investigated together with the effect of weakening the postulates on inverses and the unit. Methods of completion to form a proper group are discussed. (Received August 2, 1939.)

297. W. L. Ayres: *Peano spaces as the theory of continuous images of intervals.*

The properties of Peano spaces have been deduced almost exclusively from other characterizations rather than the original definition as the image of a closed interval under a continuous mapping. As Peano continua appear in applications usually from this definition, it seems desirable to develop the theory from this definition alone, and the present paper presents such a development. In particular a new and fairly simple proof of the important cyclic connectivity theorem is obtained, depending only on the arc-wise local connectivity of the space and arc-wise connectivity of connected open sets, both of which are proved easily from the fundamental definition. (Received July 29, 1939.)

298. Reinhold Baer: *A Galois theory of linear systems over commutative fields.*

N. Jacobson has recently succeeded in extending Galois' theory from commutative fields to non-commutative fields. A different treatment of these problems is evolved in this note. The characteristic feature of the method employed is the (elementary) proof of general reduction theorems which permit the translation of the Galois theory of any class of groups of automorphisms of commutative fields into a theory concerning groups of automorphisms of more general domains like non-commutative fields and linear systems over commutative fields. This method permits therefore, in particular a generalization of Galois' theory of infinite algebraic extensions. (Received July 25, 1939.)

299. Reinhold Baer: *Nilpotent groups and their generalizations.*

The following three properties of groups are equivalent for finite groups and may be used as definitions of nilpotent groups: (1) The group is swept out by its ascending central-series. (2) The group is a direct product of p-groups. (3) If S and $T<S$ are subgroups such that there is no subgroup between T and S, then T is a normal subgroup of S. For infinite groups the situation changes and an investigation of the interrelations of these three properties is made. A seemingly only slightly stronger property than (3) is: (4) if S and $T<S$ are subgroups, and if there exists at most one subgroup between S and T, then T is a normal subgroup of S. It may be proved that a group which satisfies (4) and which satisfies either (1) or does not contain elements of infinite order has the property that all its subgroups are normal subgroups. (Received July 25, 1939.)

300. Miriam F. Becker and Saunders MacLane: *The minimum number of generators for inseparable algebraic extensions.*

An inseparable finite extension of a field K cannot be obtained, in general, by the adjunction of a single element to K. In this paper the minimum number of generators is determined for all extensions of a fixed base field and also for specific extensions. In applying the criterion for the minimum number of generators it is necessary to compute an invariant, the degree of imperfection, of the given field. The effect of algebraic extensions, finite and infinite, upon the degree of imperfection is thus included. (Received June 27, 1939.)

301. P. O. Bell: *Projective invariants of a curve on a surface.*

Let x denote Fubini's normal coordinates for a general point of a surface S_x in ordinary space. One of the uses to which the author has applied his concept of the R_λ-associate of a line is the geometric characterization of the projective arc-length of a finite arc of a curve on S_x. The elements obtained in this characterization serve to determine geometrically the sequences of points: d^ix/ds^i and $d^ix/ds^i \pm 2d^{i-1}x/ds^{i-1}$, $(i=1, 2, \cdots, n)$, where differentiation is with respect to arc-length s along a curve C_λ of S_x. These determinations form the groundwork for geometric definitions of a number of important invariants of C_λ at x. Definitions are given for the second order invariants known as projective curvature and asymptotic curvature of a curve C_λ at x and for the third order invariant known as projective torsion. Finally a method is given for the geometric characterization of invariants of any order. (Received July 31, 1939.)

302. Stefan Bergmann and W. T. Martin: *On the existence of a function of integrable square which satisfies an infinite system of integral relations.*

The paper investigates the conditions under which there exists a (complex) function $f(t_1, t_2) \in L^2$ for $0 \leqq t_k \leqq 1$, $k=1, 2$, which satisfies a system of equations $\int_0^1\int_0^1 f(t_1, t_2)t_1^{\lambda_{1n}}t_2^{\lambda_{2n}}dt_1dt_2 = X_n$, where $\{\lambda_{1n}, \lambda_{2n}\}$, $\Re\{\lambda_{kn}\} > -\frac{1}{2}$ and $\{X_n\}$ are given systems of complex numbers. It is shown that this problem possesses a solution if and only if there exists a function $F(z_1, z_2)$ of two complex variables regular in $\Re(z_k) > 0$, satisfying $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x_1+iy_1, x_2+iy_2)|^2 dx_1dx_2 < c < \infty$ and taking on the values X_n at the points $[(\frac{1}{2}+\lambda_{1n})i, (\frac{1}{2}+\lambda_{2n})i]$. Since an infinite series can be given whose convergence is a necessary and sufficient condition for the existence of a solution of this interpolation problem (see Bergmann, Bulletin de l'Institut de Mathématiques et Mécanique à l'Université de Tomsk, vol. 1 (1935–1939), pp. 242–256, esp. p. 250), the solution is obtained explicitly when it exists. By using the theory of regions with distinguished boundary surfaces the authors treat analogous problems with other integral conditions. (Received July 28, 1939.)

303. Garrett Birkhoff: *Neutral elements in general lattices.*

An element a of a lattice is called neutral if every triple $\{a, x, y\}$ generates a distributive sublattice. It is proved that the neutral elements of any lattice L form a (distributive) sublattice, consisting of those elements carried into $[I, 0]$ under isomorphisms of L with sublattices of direct products. Further, complements of neutral elements, when they exist, are unique and neutral. The sublattice of complemented neutral elements may be called the "center" of a lattice: it consists of those elements carried into $[I, 0]$ under isomorphisms of the lattice with direct products. (Received July 17, 1939.)

304. Richard Brauer: *On groups whose order is divisible by the first power of a prime.* II. Preliminary report.

Using the results of an earlier paper (abstract 45-5-164), groups G are studied which satisfy the following conditions: (1) the order g of G is of the form $g=pg'$, where p is a prime and g' not divisible by p, (2) the elements of order p commute only with their own powers, (3) G is simple. The first two of these conditions are, for instance, satisfied for all transitive permutation groups of prime degree p, and (3) does not imply any essential restriction in this case. The order g is of the form

$g=qp(1+np)$ where $q \mid (p-1)$. It is shown that if $n<(3p+13)/4$, then either G is cyclic, or $G \simeq LF(2, p)$, or p is a prime of the form $2^h \pm 1$ and $G \simeq LF(2, 2^h)$, or $G \simeq LF(3, 3)$. Incidentally, it can be shown that $LF(3, 3)$ is the only simple group of the order 5616. (Received July 28, 1939.)

305. R. S. Burington: *On transient similarity and equivalence in linear networks.*

Let $A_i \equiv (a_{rs}^i)$, where the a_{rs}^i are real polynomials in λ, be a network matrix of order n for an n-mesh linear network N_i. N_i is said to be *transiently similar* to N_j if A_i and A_j have the same real Segre characteristic. In the present paper the concept of transient similarity is used to give a complete classification of the set of all n-mesh networks with respect to transient behavior. The case $n=2$ is considered in detail and gives a complete and improved solution of the problem undertaken by W. Quade (*Klass. der Schwing. in gekoppelten Stromkreisen*, Leipzig, 1933). The connections between steady-state equivalence, transient similarity and transient equivalence are considered in detail. (Received July 31, 1939.)

306. S. S. Cairns: *Homeomorphisms between topological manifolds and analytic manifolds.*

The theorems here stated are obtained from previously announced results (abstract 45-3-110) with the aid of methods due to Hassler Whitney (*Differentiable manifolds*, Annals of Mathematics, (2), vol. 37 (1936), pp. 645–680). A polyhedral manifold P^r in a euclidean n-space E^n is said to be *in normal position* if there exists a continuously varying $(n-r)$-plane $\pi^{n-r}(p)$ transversal to P^r at p (cf. abstract 45-3-110). Theorem I: It is possible, by introducing suitable coordinate systems to make a given topological manifold M analytic, with an analytic Riemannian metric, if and only if M is homeomorphic to some polyhedral manifold in normal position. Since every P^3 can be put in normal position, the following theorem results. Theorem II: Any topological 3-manifold which can be triangulated can be made analytic, with an analytic Riemannian metric. (Received July 31, 1939.)

307. H. H. Campaigne: *Suschkewitsch correspondences.*

Suschkewitsch (Transactions of this Society, vol. 31 (1929), pp. 204–214) has considered quasi-groups G^* obtainable from a group G by making a permutation in the headline of the Cayley multiplication table. This suggests the triple correspondence of G^* to G, where each element g^* of G^* determines three elements g_1, g_2, g_3 of G, and each element of G is determined by three of G^*, such that $a^*b^*=c^*$ if and only if $a_1b_2=c_3$. Suschkewitsch studied such a correspondence for the special case $g_1=g_3$. These correspondences form an equivalence relation among groupoids. The groupoid G^* is a quasi-group if and only if G is a quasi-group. A second correspondence, $G^* \otimes G$, is also studied. This is a one-to-one correspondence $g^* \rightarrow g$, such that $ab=c$ if and only if $a^*c^*=b^*$. The lattice of all subgroupoids of G^* is isomorphic to that of G. The group of automorphisms of G^* is isomorphic to that of G. The associative law is not preserved under either of these correspondences. (Received August 2, 1939.)

308. H. S. M. Coxeter: *A new form of coordinates for the polytope 2_{21} whose twenty-seven vertices correspond to the lines on the general cubic surface.*

The new coordinates proposed for the 27 points are the real and imaginary parts

of the complex numbers $(0, \omega^\lambda, -\omega^\mu)$, $(-\omega^\mu, 0, \omega^\lambda)$, $(\omega^\lambda, -\omega^\mu, 0)$, where ω^λ, $(\lambda = 0, 1, 2)$, are the cube roots of unity. These complex numbers are the first three coefficients in the equations of the 27 linear complexes "of the first kind" which are permuted among themselves by the quaternary collineation group of order 25920 (Burkhardt, Mathematische Annalen, vol. 41 (1892), p. 323). The remaining three coefficients are their respective conjugates. When 0, 1, ω, ω^2 are interpreted as the marks of the field $GF[2^2]$, the collineation group becomes $HO(4, 2^2)$, and the linear complexes become special, so that they can be replaced by single lines of the finite geometry $PG(3, 4)$ (Frame, this Bulletin, vol. 44 (1938), p. 659). These are, in fact, the 27 lines on the "cubic surface" whose 45 points satisfy the congruence $x_0^3+x_1^3+x_2^3+x_3^3 \equiv 0$ in the Galois field. The remaining 40 points of the finite geometry are found to correspond to 40 concurrent planes of euclidean eight-space which together contain the 240 vertices of the polytope 4_{21}. (Received July 25, 1939.)

309. Bernard Dimsdale: *Degree of approximation by linear combinations of powers.*

Szász (Mathematische Annalen, vol. 77 (1916), pp. 478–493) has shown that any continuous function may be uniformly approximated by a linear combination of powers for which the exponents satisfy certain conditions. The present paper, based on a part of the work of Szász, contains a corresponding discussion of degree of approximation. With certain further restrictions on the sequence of exponents, results are obtained which are comparable in form and generality with those that hold for approximation by ordinary polynomials in successive integral powers of the variable. (Received July 31, 1939.)

310. J. E. Eaton: *Theory of cogroups.*

A cogroup is a special multigroup in which every submultigroup is reversible. Since the coset decomposition of any group with respect to a subgroup is a cogroup, a few of the results are of group theoretical interest. (Received July 24, 1939.)

311. Benjamin Epstein: *Growth properties of analytic functions of two complex variables.*

Using the theory of domains with distinguished boundary surfaces (ausgezeichnete Randflächen) recently established by S. Bergmann, the author obtains theorems concerning the growth in certain non-analytic three-dimensional hypersurfaces $\mathfrak{S}^3$ of analytic functions $f(z_1, z_2)$, $z_k = x_k + iy_k$, $k = 1, 2$, regular in $\mathfrak{M}^4$: $x_k \geqq 0$. For instance, if $|f(iy_1, iy_2)|$ is bounded for $-\infty \leqq y_k \leqq +\infty$ and if $|f(iy_k, z_k)| \leqq C|r_k|^{1-\epsilon} + |y_k|^{\alpha_k}$ (where α_k can be explicitly found), then either $|f(z_1, z_2)| \leqq 1$ in $\mathfrak{M}^4$ or $\max |f(z_1, z_2)|$ must exceed a certain lower limit, in every $\mathfrak{S}^3$, for its growth. Furthermore theorems are obtained about zero-lines in these hypersurfaces $\mathfrak{S}^3$. Analogous results are obtained for the case in which $\mathfrak{M}^4$ is $|z_1| < \infty$, $-\pi/2k_1 \leqq \arg z_1 \leqq \pi/2k_1$, $|z_2| < \infty$, $-\pi/2k_2 \leqq \arg z_2 \leqq \pi/2k_2$. (Received July 26, 1939.)

312. F. A. Ficken: *Cones and vector spaces.*

This paper is concerned with the relation of an arbitrary vector space V to a quadric cone C in a real vector or centered affine n-space. By exploiting the contact-space of a V tangent to C, and using the most elementary concepts of lattice theory, a simple, complete, and highly intuitive geometric account is given of the intersection of V with C. The argument yields many known results, and leads to a classification

of vector spaces which is shown to be the equivalence-classification under the Lorentz group of transformations which leave C fixed. (Received July 31, 1939.)

313. Bernard Friedman: *Fourier coefficients of bounded functions.*

Let $f(x)$ be a bounded measurable function in $(-\pi, \pi)$ such that $|f(x)| \leqq 1$. Let a_n, b_n be the Fourier coefficients of $f(x)$ so that $\pi a_n = \int_{-\pi}^{\pi} f(x) \cos nx\, dx$, $\pi b_n = \int_{-\pi}^{\pi} f(x) \sin nx\, dx$; $n = 1, 2, \cdots$. Obviously $|a_n|$, $|b_n| \leqq 4/\pi$. This paper shows that these coefficients must lie in definite regions. For example, b_1 and b_2 must lie in the following region: $-16+\pi^2 b_1^2 \leqq 4\pi b_2 \leqq 16-\pi^2 b_1^2$. The above inequalities are the best possible in the sense that if a set of a, b make the inequality an equality, there exists a bounded function having those numbers as its Fourier coefficients. Also, for any bounded $f(x)$ and any integer n, there exists a function $g_n(x)$ taking on only the values $+1$ and -1 such that the first n Fourier coefficients of $f(x)$ and $g_n(x)$ are the same. Incidently there is proved the following result: If $F(x)$ is a bounded measurable function such that $|F(x)| \leqq \frac{1}{2}$ then $\int_0^{\pi} F(x) \cos x dx \leqq \cos\left(\int_0^{\pi} F(x) dx\right)$. (Received August 2, 1939.)

314. B. E. Gatewood: *Thermal stresses in long cylindrical bodies.*

The problem of finding thermal stresses and displacements in long cylinders is reduced to obtaining a particular integral for the Poisson equation $\nabla^2 V = kT$, where T is the temperature distribution and k is a constant, and to solving the biharmonic equation $\nabla^4 U = 0$ when the first derivatives of the function U are known on the boundary of the crosssection of the body. These two differential equations are related by the condition that on the boundary the first derivatives of U are given in terms of the first derivatives of V. The biharmonic equation is solved by use of two analytic functions (Muschelisvili, Bulletin de l'Académie des Sciences de Russie, vol. 13 (1919); Mathematische Annalen, vol. 107 (1933)). The region within which the solution is desired is mapped conformally on the unit circle by a rational function, and from the boundary values on the unit circle the required analytic functions are determined directly by the use of the Cauchy integral formula. The methods are extended to multi-connected bodies and to bodies composed of several different materials. Special problems of the circular cylinder, the hollow circular cylinder, the concentric and eccentric composite circular cylinders are solved. (Received July 24, 1939.)

315. Michael Goldberg: *New five-bar and six-bar linkages in three dimensions.*

A closed chain of n hinged links in three dimensions has, in general, $n-6$ degrees of freedom. When n is less than 7 the chain is generally rigid; a movable chain is said to be "paradoxical." Bricard (*Leçons de Cinématique*, Paris, 1927) lists all the known paradoxical chains. His list contains chains of four links and of six links, but it does not contain any chains of five links. This paper describes the structure and motion of a new family of chain linkages of five links, and also, two new types of six links. (Received July 21, 1939.)

316. L. M. Graves: *On the fundamental lemma of Haar in the calculus of variations.*

A "classical form" of fundamental lemma for double integrals may be stated as follows: (a) let the functions $U(x, y)$, $V(x, y)$, $W(x, y)$, $\partial U/\partial x$, $\partial V/\partial y$ be continuous on a bounded open set A; and (b) let $\iint_A [U\xi_x + V\xi_y + W\xi] dx\, dy = 0$ for every ξ of

class C' vanishing on the boundary B of A; then (c) $W=\partial U/\partial x+\partial V/\partial y$ on A. Let $U_\epsilon(x, y)$, $V_\epsilon(x, y)$, $W_\epsilon(x, y)$ be the integral means of U, V, W respectively, taken over the square $S(x, y, \epsilon)$ with center (x, y) and side 2ϵ. Then when the hypothesis (a) is replaced by: (a_1) let the functions U and V be bounded and measurable and W be integrable over A; the conclusion (c) may be replaced by: (c_1) for every (x, y, ϵ) such that $S(x, y, \epsilon)$ is in A, the partial derivatives $\partial U_\epsilon/\partial x$ and $\partial V_\epsilon/\partial y$ exist and $W_\epsilon=\partial U_\epsilon/\partial x+\partial V_\epsilon/\partial x$. Previous proofs of a related form of the Haar fundamental lemma under the weak hypothesis (a_1) have obtained an equation holding only for almost all rectangles. The author also shows that when (a_1) is replaced by the somewhat stronger: (a_2) let the functions U and V be continuous on A and let W be integrable over A; then (c_1) may be obtained very simply as a consequence of the "classical form" of the fundamental lemma stated above. (Received July 27, 1939.)

317. L. M. Graves: *Some general approximation theorems.*

In this paper comparatively simple proofs are given for some general theorems on the approximation of a given function by "smoother" functions. Well known theorems on approximation by polynomials, by trigonometric sums, and by integral means are special cases. The proofs are formulated for functions of any finite number of variables. Theorems on the term-by-term differentiation and integration of the approximating sequence are included. When the given function is continuous, and vanishes on a closed point set E, it is shown that the approximating functions may be required to vanish on a neighborhood of E while possessing as many continuous derivatives as may be desired. (Received July 22, 1939.)

318. D. W. Hall (National Research Fellow): *On pointwise periodic homeomorphisms in the plane.*

Let M be a continuous curve on the 2-sphere S_2 which is not separated by the omission of any pair of points. Let $T(M)=M$ be pointwise periodic. Then T is periodic and may be extended to a periodic transformation on S_2 and hence must be identical with an elementary euclidean transformation. As a result the negative solution to a problem of W. L. Ayres (see abstract 45-1-87) is obtained. (Received July 28, 1939.)

319. D. W. Hall (National Research Fellow) and J. L. Kelley: *On continuous collections of orbits.*

Let M be a compact metric space and $T(M)\subset M$ be a homeomorphism. In order that the collection of point orbits in M under T form a continuous collection, each of the following conditions is necessary and sufficient: (1) if K is a closed set in M, then K together with all its images forms a closed set; (2) T is pointwise periodic and almost periodic, that is, every point orbit is finite and for $\epsilon>0$ there exists a positive integer L, such that in every set of L consecutive integers there is an n such that $\rho[x, T^n(x)]<\epsilon$ for all $x \in M$. (Received July 28, 1939.)

320. P. R. Halmos: *On a necessary condition for the strong law of large numbers.*

Let $\{x_n\}$ be a sequence of independent chance variables, and $\bar{x}_n=x_n$ when $|x_n|\leqq n$, $\bar{x}_n=0$ otherwise. Write $b_n=E[\bar{x}_n-E(\bar{x}_n)]^2$, and suppose that $1/n\sum_1^n E(\bar{x}_j)\to 0$. The following theorem (known as the strong law of large numbers) is an easy consequence of a result due to Kolmogoroff: If $x_n/n\to 0$ with probability 1 and if $\sum_1^\infty b_n/n^2<\infty$,

then $1/n\sum_1^n x_j \to 0$ with probability 1. The converse of this theorem is not known and it is the purpose of this paper to make a contribution in this direction by proving the following theorem: If $1/n\sum_1^n x_j \to 0$ with probability 1, then $x_n/n \to 0$ with probability 1, and $\sum_1^\infty b_n/n^{2+\epsilon} < \infty$ for all $\epsilon > 0$. (Received July 20, 1939.)

321. O. G. Harrold: *Topological properties of rectifiable continua.*

It is shown in this paper that the property of a compact continuum of being a topological image of a rectifiable continuous curve is cyclicly extensible and reducible. If M is a compact continuum of finite degree (Kamiya), there exists a topological image M^1 of M and a continuous transformation T of the unit interval I such that $T(I) = M^1$ and the length of M^1 with respect to T is finite. If M is any rectifiable continuous curve, M is of finite degree. (Received August 3, 1939.)

322. Philip Hartman and R. B. Kershner: *On some limit theorems for number theoretic functions.*

In the study of number theoretic functions various authors have proved a number of limit results of which the following is typical: $\liminf \phi(n) \log \log n/n = e^{-\gamma}$ where $\phi(n)$ is the Euler ϕ-function and γ is the Euler-Mascheroni constant. In this paper several general theorems of this type are proved by very elementary methods. These general theorems include all of the classical results as special cases. (Received August 1, 1939.)

323. Olaf Helmer: *The principal ideal theorem in domains of integral functions.*

Let $K\langle z\rangle$ denote the domain of integral functions $f(z) = \sum a_n z^n$ with coefficients a_n in K, K being an arbitrary number field. P may stand for the field of rational numbers. Every $K\langle z\rangle$ is an integral domain, and one can speak of the ideals in $K\langle z\rangle$. This paper is concerned with the question as to whether there are any but principal ideals in $K\langle z\rangle$. The answer depends upon the precise definition of "ideal" that is employed. In contradistinction to the classical definition, Prüfer (Journal für die reine und angewandte Mathematik, vol. 168 (1932)) introduced a narrower ideal concept which comprises only the ideals (in the classical sense) with a finite basis. And the answer can now be formulated as follows. A: Every $K\langle z\rangle$ contains ideals in the classical sense which are not principal ideals. B: If only ideals in the Prüfer sense are admitted, then every ideal in $K\langle z\rangle$ is a principal ideal. (Received July 28, 1939.)

324. T. R. Hollcroft: *Anomalous systems of primals.*

An anomalous system of primals is a system for which the invariant effective dimension is greater than the invariant virtual dimension. In 1929, B. Segre discovered an anomalous (*irregolare*) system of plane curves with dependent cuspidal invariants. In the present paper it is shown that anomalous systems of primals exist with various types of distinct double points. In particular, the system $\sum f_i^2 = 0$ ($i = 1, \cdots, r$, each f_i being a nonsingular primal of order α in S_r) has α^r nodes lying at the intersections of the ∞^{r-1} linear system $\sum \lambda_i f_i = 0$. A general expression for the anomaly of this system is found. For $r = 3$, this is a system of surfaces of order α with α^3 nodes lying at the intersections of a net of surfaces of order α. The anomaly of this system is $(\alpha-1)(\alpha-2)(\alpha-3)/6$. (Received July 27, 1939.)

325. D. H. Hyers: *A generalization of Fréchet's differential.*

Several definitions of differentials in linear topological spaces have already been given. (See A. D. Michal and E. W. Paxson, Comptes Rendus, Warsaw Academy of Sciences, vol. 29 (1936), pp. 106–121, and A. D. Michal, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 340–342.) In the present note the differential of Fréchet is generalized by making use of pseudo-norms in place of norms. The definition of this "*F*-differential" applies to functions with arguments in any linear topological space and values in any other linear topological space. The usual rule for the differentiation of iterated functions is shown to hold. Every *F*-differentiable function is *M*-differentiable in the sense of Michal. When the linear topological spaces can be normed, the *F*-differential coincides with the Fréchet differential. It is still an open question whether this is also true of the *M*-differential. (Received July 25, 1939.)

326. Dunham Jackson: *Orthogonal polynomials with auxiliary conditions.*

In recent papers before section meetings of the Association the writer has presented examples of systems of orthogonal polynomials subject to linear homogeneous auxiliary conditions. The conclusions noted in those papers for the special cases considered there are now extended to form a theory of some degree of generality, at least on the formal side, while the theory of convergence also is carried somewhat beyond the stage previously reached. (Received July 31, 1939.)

327. Fritz John: *Special solutions of certain difference equations.*

It was proved by H. Bohr, that any continuous, convex solution of the difference equation $y(x+1)-y(x)=\log x$ differs from $\log \Gamma(x)$ by a constant. Recently A. Meyer proved (Acta Mathematica, vol. 70, p. 59), that the functional equation $y(x+1)=1/x\cdot y(x)$ has only one convex solution. In this paper the author proves theorems of this type for more general difference equations, for example: Every two continuous, convex solutions of the difference equation $\sum_{i=0}^{n} a_i f(x+i)=g(x)$ with constant coefficients differ at most by a constant, if (1) all roots of the characteristic equation are simple and of absolute value 1, (2) $g(x)$ is continuously differentiable, (3) either $\lim_{x\to\infty} g'(x)=0$, or $\sum_i a_i=0$ and $\lim_{x\to\infty} g'(x)/x=0$. (Received July 17, 1939.)

328. Wilfred Kaplan: *Regular curve-families filling the plane.*

Let a family F of curves in the plane be given such that (a) each curve is the homeomorphic image of a circle or of an open interval, (b) through each point of the plane passes one and only one curve, (c) locally the family is homeomorphic with a set of parallel lines. A family of this type can be obtained as the level curve family of a continuous function $f(x, y)$. This paper establishes the converse: every such family F is the level-curve family of a non-trivial continuous function $f(x, y)$. This is proved by means of a "normal" subdivision of the family F into nonoverlapping pieces, in each of which the family is homeomorphic with the parallel lines of a half-plane. (Received July 28, 1939.)

329. Edward Kasner and J. J. De Cicco: *Trihornometry of second order.*

This paper is a continuation of the paper by the authors, *Conformal geometry of horn angles of second order* (Proceedings of the National Academy of Sciences, vol. 24

(1939), pp. 393–400). The laws connecting the conformal invariants of a trihorn of second order are obtained. This configuration consists of three curves (C_1, C_2, C_3) which pass through a common point in a common direction such that all three curves have the same curvature but no two of them have the same rate of variation of curvature at the common point. A trihorn possesses nine conformal invariants: the three sides or measures M_{12}, M_{23}, M_{31}, the three direct dihorn angles α_{12}, α_{23}, α_{31}, and the three reverse dihorn angles α_{21}, α_{32}, α_{13}. The laws connecting these nine parts are analogous to the angular identity, and the laws of sines and cosines of ordinary plane geometry. In general, four parts of a trihorn have to be given in order to determine the remaining five parts. (Received July 20, 1939.)

330. W. S. Kimball: *Partial derivatives of derivatives.*

The parallel displacement method used previously with line integrals and the calculus of variations is here extended to include second and higher partial derivatives where the slope y' appears in the vector integrand. The need for partial derivatives of the slope requires a new definition of the derivative admitting parallel displacement differentiation: The derivative of y with respect to x is the quotient of $dy_1 = Ln$ by $dx = \Delta x = n$, where n represents unit positive magnitude, and L is the limit heretofore defined as the derivative. The new derivative equals the same limit as formerly and involves no new departure except that it particularizes the differentials whereby dx for the independent variable must be unit positive magnitude n, and with corresponding restriction on dy. The rate concept using parallel displacements leads uniquely to the new partial derivatives, which are functions of units in contrast with classical derivatives. These partial derivatives of y' are required by mathematical consistency to fill a gap left by classical differentiation when length elements as well as density vary along a wire. (Received July 31, 1939.)

331. M. S. Knebelman: *Scalar invariants of the vector vortex excess.*

In a recent paper (American Journal of Mathematics, vol. 61, pp. 461–472), Hermann Weyl introduces the scalar invariants H_e defined over a Riemann space C_ν embedded in a spherical space of curvature $1/\nu^2$ where e is even and $0 \leqq e \leqq \nu$. In this note it is shown that, except for a numerical factor, H_e is the trace of the $e/2$th compound of the matrix $H_{\alpha\beta}^{\lambda\mu}$, where the pairs of different indices are arranged in a certain order, and that the invariants H_i for $i = 1, 2, \cdots, \nu(\nu-1)/2$ form a complete set of scalar invariants. (Received July 3, 1939.)

332. R. E. Langer: *On the stability of the laminar flow of a viscous fluid.*

The effect of the disturbance in the form of an elementary wave of length $2\pi/\alpha$ upon the flow of a fluid between parallel plane boundaries has been variously studied by other authors. With customary assumptions of a physical nature the problem may be made to depend upon the solution of an ordinary differential equation of the fourth order subject to four boundary conditions. The coefficients of this equation involve the wave length of the disturbance, the Reynold's number of the fluid, R, and a parameter C. For characteristic values of the latter the problem is solvable. Depending on whether the imaginary part of any characteristic value is positive or negative, the associated motion of the fluid is stable or unstable. This paper discusses the asymptotic nature of the configuration when αRC is large. It is shown that all characteristic values lie in the strip of the complex plane $0 < \Re(C) < 1$, $0 < \Im(C)$ augmented

by a small neighborhood of the real interval from 0 to 1. Thus, except possibly for some particular motions near the borderline of turbulence, other motions of the fluid are shown to be stable. (Received July 31, 1939.)

333. C. G. Latimer: *A class of diophantine equations.*

Let α, β be rational integers such that (a) one is negative, (b) not both are in the form $4n+3$ and (c) $\alpha\cdot\beta$ is not divisible by a square greater than 1. $f\equiv x^2+\alpha y^2+\beta z^2+\alpha\beta w^2$ is the norm of the general element in a ring $\mathfrak{G}$ of integral elements in a quaternion algebra $\mathfrak{A}$. If d is the fundamental number of $\mathfrak{A}$ it is known that $\Delta\equiv 2\alpha\cdot\beta/d$ is an integer. Let $\Delta=2^{\epsilon}\cdot\delta$ where $\epsilon=0$ or 1 according as Δ is odd or even. In this paper we determine all solutions in integers of $f=u_1\cdot u_2\cdot\cdots\cdot u_n$, where the u's are prime to δ. This result is obtained after showing that every one-sided ideal in $\mathfrak{G}$ which contains a rational integer prime to δ is a principal ideal. For the case where $\alpha=1$, $\beta=-3$ our result is the same as that obtained by Dickson (*Algebren und ihre Zahlentheorie*, p. 173, Theorem 13). Similar results may be obtained for the equation $f_1=u_1\cdot u_2\cdot\cdots\cdot u_n$ where f_1 is any form which may be transformed into f by a transformation with integral coefficients. (Received July 31, 1939.)

334. D. H. Lehmer: *Certain congruential properties of spherical harmonics.*

Let $P_n(x)$ be Legendre's polynomial and let $Q_n(x)$ be the second solution of his differential equation. Finally let $Z_n(x)$ denote the familiar polynomial of degree $n-1$ defined by $P_n(x)\left[\log(x+1)-\log(x-1)\right]-2Q_n(x)$, frequently designated by $2W_{n-1}(x)$. Several identical congruences connecting P, Q, and Z are derived in this note of which the following are typical: (1) $Z_{p-1}(x)\equiv 0 \pmod p$ identically in x; (2) $pZ_{p+r}(x)\equiv 2P_r(x) \pmod p$. In both cases p is a prime. (Received July 25, 1939.)

335. Jakob Levitski: *On multiplicative systems.*

In the present note it is proved that each nil-ring, in a ring which satisfies a certain maximum condition, is nilpotent. This is an extension of a result first found by Charles Hopkins (Duke Mathematical Journal, vol. 4 (1938), pp. 664–667) concerning rings which satisfy a certain minimum condition. In particular follows the theorem: Each right or left nil-ideal of a ring which satisfies the minimum or the maximum condition for the right ideals, is nilpotent. This solves a problem proposed by Koethe. The results obtained are based upon two general theorems, which are possibly of independent interest, on multiplicative nil-systems. (Received July 21, 1939.)

336. D. C. Lewis: *The separability of Pfaffian transformations.*

A nonsingular analytic transformation T of a region of n dimensional euclidean space into another such region is called Pfaffian with respect to a linear differential form ω, if it admits $\int\omega$ as a relatively invariant line integral. If coordinates $x_1,\cdots,x_n$ are introduced in such a manner that ω can be written in the form ω' plus an exact differential, where ω' depends only on $x_1,\cdots,x_{2k}$, $(2k<n)$, then T takes the form: $\bar{x}_i=f_i(x_1,\cdots,x_{2k})$, $i=1,\cdots,2k$; $\bar{x}_j=g_j(x_1,\cdots,x_n)$, $j=2k+1,\cdots,n$; provided that ω' cannot be expressed in fewer than $2k$ variables. This result is closely analogous to a classical result of Lie and Koenig for differential equations. Applications are made to Pfaffian transformations admitting invariant manifolds. (Received July 17, 1939.)

337. Ingo Maddaus: *On types of "weak" convergence in normed vector spaces.* Preliminary report.

This paper is concerned with the study of weak convergence and convergence which is weaker than weak convergence. The fundamental notions, with postulational restrictions, are (1) an association (x, y) to any pair x and y of elements of the space, which is of the nature of a generalized "metric," the range not being restricted to real numbers, (2) a real number or norm corresponding to every finite number of elements and to every finite number of associations of (1). A limit notion, $x_n \to_H x$, is defined by means of the norm of a finite number of associations of (1). In the space of all continuous functions on $0 \leqq t \leqq 1$, $x_n \to_H x$ is equivalent to $x_n(t) \to x(t)$ for each t. Realizations of $x_n \to_H x$ are given for (l^p), (L^p), $(p \geqq 1)$, and for (c). By different interpretations of (1) and (2), $x_n \to_H x$ is found to be equivalent to weak convergence and to a still weaker convergence in (l^p), (L^p), $(p \geqq 1)$. Properties of linear, H-continuous functionals and transformations are studied and general forms are obtained. (Received July 31, 1939.)

338. I. Malkin: *On the elastic problem of the plate.* Preliminary report.

A solution of the well-known Michell type of the elastic problem of the plate is obtained explicitly by using an operational expansion of the biharmonic function of the space in the form $u = u_0 + \sum (-1)^{n+1}(z^{2n}/(2n)!)D^{n-1}(nU_0 - Du_0) + (z/1!)u_1 + \sum (-1)^{n+1}(z^{2n+1}/(2n+1)!)D^{n-1}(nU_1 - Du_1)$ wherein u_0, u_1, U_0, U_1 are functions of x and y, and D is the plane Laplacian operator, the displacement components being biharmonic functions in the case of vanishing body forces. With the aid of such expansions for the displacement components the solution of the elastic problem of the plate is expressed in terms of six arbitrary functions of x and y, which suffice to satisfy the boundary conditions of the plane surfaces of the plate, while on the cylindrical surface they are fulfilled in using the principle of de St. Venant. In the case of a plate under normal load the problem is reduced to determination of two arbitrary functions. This case is worked out in detail in the paper and illustrated by a numerical example in finite polynomials. In connection with previous publications on analogous solutions of the harmonic problem (conduction of heat), the present paper is intended as a basis for the treatment of the thermoelastic problem of plates and shells. (Received July 28, 1939.)

339. N. H. McCoy: *A generalization of Ostrowski's theorem on matric identities.*

Recently A. Ostrowski (Quarterly Journal of Mathematics, vol. 9 (1938), pp. 241–245) has proved a theorem which is, in a certain sense, an analogue for several matrices of Frobenius' theorem on the minimum equation of a matrix. In the present note, one part of Ostrowski's theorem is generalized to the case in which the elements of the matrices are elements of an arbitrary commutative ring R with unit element. By making one additional restriction on R, the other part of the theorem is similarly generalized. (Received August 4, 1939.)

340. E. J. McShane: *An estimate for the Weierstrass $\mathcal{E}$-function.*

It is known that if a curve C: $y = y(x)$, $a \leqq x \leqq b$, satisfies the differential equations of a Bolza problem and is nonsingular, and along it the strengthened

Weierstrass condition Π'_N holds, then for all (x, y) near C an inequality of the type $\mathcal{E}(x, y, y', Y', \lambda) \geqq \delta(\epsilon)\|y'-Y'\|$, $(\delta(\epsilon)>0)$, holds whenever (x, y, y') and (x, y', Y') are admissible and $\|y'-Y'\| \geqq \epsilon$. Here it is shown that the hypothesis of nonsingularity can be discarded. Incidentally, it is proved that even for singular curves the transformation to canonical variables is topological, provided that the strengthened Weierstrass condition is satisfied. (Received July 28, 1939.)

341. E. J. McShane: *A remark concerning sufficiency theorems for the problem of Bolza.*

By virtue of recent theorems of Hestenes, Morse and Reid, a curve $y=y(x)$ is known to be a minimizing curve for a Bolza problem if there are multipliers $\lambda_0=1$, $\lambda_\alpha(x)$ with which it is an extremal and satisfies the strengthened Weierstrass, Clebsch and Jacobi conditions. Proofs of necessary conditions yield only that there are multipliers $\lambda_0 \geqq 0$, $\lambda_\alpha(x)$ with which the Euler, transversality, Weierstrass and Clebsch conditions hold. Here it is shown that the discrepancy concerning λ_0 is easily removed; the sufficiency theorem remains valid if the hypothesis $\lambda_0=1$ is replaced by $\lambda_0 \geqq 0$. (Received July 28, 1939.)

342. E. J. McShane: *Generalized curves.*

A generalized curve in the sense of L. C. Young is a system consisting of a set of functions $y=y(t)$, $a \leqq t \leqq b$ (where y is a vector in ν-space), and a linear average $\mathcal{M}[t; \Phi(r)]$ defined on the space of continuous functions $\Phi(r)$ (r also is a vector in ν-space); there are of course certain restrictions on the $y(t)$ and the $\mathcal{M}$. Ordinary curves are special cases obtained by choosing $\mathcal{M}[t; \Phi(r)] = \Phi(y'(t))$. In the present paper Young's theory of these curves is presented and extended. The space of generalized curves can be so metrized that every integral of the calculus of variations is continuous in the metric, and every bounded closed set in the space of curves is compact. This leads at once to existence theorems concerning the existence of minimizing generalized curves for generalized-curve problems of Bolza type. (Received July 28, 1939.)

343. E. J. McShane: *Necessary conditions for a minimum in generalized-curve problems of the calculus of variations.*

The problem studied is that of minimizing a functional (integral plus function of end points) in the class of generalized curves (see preceding abstract) which satisfy certain differential equations and end conditions. Only parametric problems are considered. Conditions are established which are necessary in order that a generalized curve afford a strong relative minimum for the problem. Analogues of the multiplier rule, transversality conditions, Weierstrass condition, Clebsch condition and Dresden corner condition are established. (Received July 28, 1939.)

344. E. J. McShane: *Existence theorems for Bolza problems in the calculus of variations.*

For generalized-curve problems of Bolza it is easy to establish existence theorems (see preceding abstract 44-9-342). By use of the necessary conditions satisfied by such a minimizing curve, it can be shown that under suitable assumptions on the data of the problem this minimizing generalized curve is actually a curve in the usual sense. The theorems thus obtained are of considerable generality, and include at least interesting subcases of all existence theorems for single-integral problems known to the author. (Received July 28, 1939.)

345. A. D. Michal: *A differential in linear topological spaces.*

A new differential, a slight variant of the M-differential, is defined for functions with arguments and values in linear topological spaces. It is equivalent to the Fréchet differential whenever the linear topological spaces are Banach spaces. (Received August 1, 1939.)

346. A. D. Michal: *Fréchet differentials and M-differentials in Banach spaces.*

This paper characterizes a Fréchet differential in Banach spaces as a special linear Gateaux differential. If the linear topological spaces are specialized to Banach spaces, then the above result is used to characterize a Fréchet differential as a special M-differential. (Received August 1, 1939.)

347. A. D. Michal, R. Davis, and Max Wyman: *Polygenic functions in general analysis.*

Analytic functions in general analysis have been studied by R. S. Martin, Michal, and more recently by A. E. Taylor and others. In this paper, the authors begin the study of polygenic functions in general analysis. The directional Gateaux differential of a polygenic function is characterized for a certain large class of polygenic functions. (Received August 1, 1939.)

348. Deane Montgomery and Leo Zippin: *Topological transformation groups.*

This paper studies the action of a compact group G of homeomorphisms of a metrizable space E. The points of E which are on orbits of dimension greater than or equal to k form an open set. The local structure of an orbit of finite dimension is similar to the local structure of a topological group of the same dimension. If G acts effectively (only the identity leaving every point fixed) on a finite dimensional orbit, then G is itself finite dimensional; if G acts effectively in a locally euclidean space, G is finite dimensional. In this case, furthermore, if all orbits are locally connected, the group G is a Lie group. (Received July 17, 1939.)

349. Deane Montgomery and Leo Zippin: *Transformation groups characterizing certain three-manifolds.*

This paper deals with a space E on which there acts a compact connected transformation group G. The space E is subjected to various restrictions, chiefly that it be locally connected, separated by certain two-dimensional manifolds (if it contains them) and "barred off" locally into precisely two sides by any two-cell it may contain. Assuming further that G has at least one two-dimensional orbit, the authors show that the space E must be one of a very few simple types. In particular it is locally three-euclidean excepting, possibly, at the points of at most two orbits. The group G must be a Lie group: in fact, one of three. The paper is preliminary to a topological group characterization of three-space and its "rigid motions" paralleling Hilbert's work in the plane. Incidentally it is shown that the only two manifolds which can be coset spaces of a compact group are the sphere, the projective plane and the torus. (Received July 26, 1939.)

350. C. B. Morrey: *The problem of Plateau on a Riemannian manifold.* Preliminary report.

Let $\mathfrak{M}$ be a Riemannian manifold of class C'. It is assumed further that each point P of $\mathfrak{M}$ is interior to a neighborhood which possesses a map of class C' on the unit cube $R: |x_i| < 1$ in which P corresponds to the origin and such that the $g_{ij}(x_1, \cdots, x_N)$ are continuous on R and satisfy the condition $m \cdot \sum_{i=1}^{N} \xi_i^2 \leqq \sum_{i,j=1}^{N} g_{ij}(x_1, \cdots, x_N)\xi_i\xi_j \leqq M \cdot \sum_{i=1}^{N} \xi_i^2$ for each $(x_1, \cdots, x_N)$ in R and each $(\xi_1, \cdots, \xi_N)$, m and M being independent of P, $0 < m < M$. Let C be a simple closed curve bounding a surface of finite area in $\mathfrak{M}$. Then there exists a surface S in $\mathfrak{M}$ of least area bounded by C which possesses a (continuous) generalized conformal map on the closed unit circle $\bar{\Sigma}$. If $\mathfrak{M}$ is of class $C_\alpha^{(n+1)}$ (that is, if the $(n+1)$st derivatives in the transformations between overlapping coordinate systems satisfy Hölder conditions with exponent α) and if the g_{ij} are of class $C_\alpha^{(n)}$, then the functions occurring in a generalized conformal map of S are of class $C_\alpha^{(n)}$ on the interior Σ of $\bar{\Sigma}$; if $\mathfrak{M}$ and the g_{ij} are analytic, these functions are analytic. (Received July 28, 1939.)

351. David Moskovitz and L. L. Dines: *On the supporting-plane property of a convex body.*

This paper extends the results of an earlier paper (abstract 44-9-361) wherein the authors considered certain questions related to the notion of *convexity* in a linear space in which a metric and the notion of a hyperplane were introduced by means of a postulated inner product. The principal result of this paper is that in such a space a set which is closed, linearly connected, and which possesses inner points is completely supported at its boundary. This result together with results obtained in the earlier paper enable the authors to write the symmetrical theorem: If a subset of elements in the space considered is closed and possesses inner points, then the properties of being (i) *linearly connected* and (ii) *completely supported* at the boundary are logically equivalent. As in the earlier paper, the space is assumed to be complete but not assumed to be separable. (Received July 24, 1939.)

352. H. T. Muhly: *Valuations and infinitely near algebraic loci.* II. Preliminary report.

In this paper the results announced by the author in abstract 45-5-216 are extended to arbitrary algebraic varieties. The base condition imposed by a simple subvariety V_{r-2} of a V_r and by varieties $V_{r-2}^{(\alpha)}$ "infinitely near" V_{r-2} are interpreted arithmetically and studied from this point of view. Any valuation B of the field Σ of rational functions on V_r which has origin at V_{r-2} defines a succession $\{V_{r-2}^{(\alpha)}\}$ of varieties infinitely near to and in the successive neighborhoods of the proper variety V_{r-2}. The author studies a class of birational transformations of V_r which resolve the varieties of the various neighborhoods of V_{r-2} into proper varieties. These transformations are a natural generalization of the quadratic plane transformations and the monoidal space transformations. The results obtained respond to the geometric expectation which leads to their formulation, and they constitute a generalization of the classical theory of infinitely near points. (Received July 31, 1939.)

353. Albert Neuhaus: *On products of normal semi-fields.*

Let S of order n over K be a direct sum of t equivalent normal fields over K, so that S has a group of automorphisms G of order n which is transitive with respect to

the t normal fields. Define a normal system (S, G) of all pairs S', G' with S' equivalent to S under a correspondence such that the elements of G' go into those of G. For cyclic G such systems were considered by A. A. Albert (Annals of Mathematics, (2), vol. 39 (1938), pp. 669–682). This paper generalizes his results for more general groups and considers crossed products of normal systems of semi-fields with their groups. (Received August 2, 1939.)

354. E. N. Oberg: *A note on a certain phase of best approximation.*

If $f, \phi_1, \phi_2, \cdots, \phi_n$ are linearly independent elements of a space having a norm which satisfies certain general hypotheses, and if $T(u)=f$ represents a linear functional equation whose elements are in this space, then it is shown that there exists a linear combination of the functions $\phi_1, \phi_2, \cdots, \phi_n$ which minimizes the norm of $f-T(\Phi_n)$, and that this minimizing function is in a certain sense an approximate solution of $T(u)=f$ when this equation has a unique reciprocal. One of the main purposes of this paper is to investigate the degree of approximation of the minimizing function in a subspace of the original space, an example of which is the study of uniform convergence of the polynomial determined by least mth power methods. (Received August 2, 1939.)

355. Rufus Oldenburger: *Infinite powers of matrices and characteristic roots.*

Hardy Cross introduced a method of balancing structures in civil engineering, where this method is an infinite process which (it can be shown) converges if and only if the infinite power of a real matrix exists and is zero. In the present paper necessary and sufficient conditions for the existence of the infinite power of a real or complex matrix are given in terms of characteristic roots and elementary divisors associated with this matrix. From these results it follows that a "greater than or equal to" relation can be defined for real matrices A and B so that if A and B are real matrices with nonnegative elements, and A is greater than or equal to B, the maximum absolute value of the characteristic roots of A is greater than this maximum for B. (Received July 10, 1939.)

356. L. F. Ollmann: *On cubic ring graphs.*

Cubic ring graphs were defined by E. W. Miller (see abstract 44-5-238). In this paper it is shown that any cubic graph may be imbedded in a cubic ring graph. Theorems concerning the joining of the vertices of a cubic ring graph (or a graph related to a cubic ring graph) by simple closed curves are also proved. Examples of cubic graphs are given in which, for any given number m, the vertices cannot be joined by m simple closed curves. (Received July 24, 1939.)

357. Isaac Opatowski: *Differential equations containing functionals of unknown functions.*

Application of the general implicit function theorem (T. H. Hildebrandt and L. M. Graves, Transactions of this Society, vol. 26 (1924), pp. 127–153; L. M. Graves, this Bulletin, vol. 41 (1935), p. 654; and L. Kantorovitch, Acta Mathematica, vol. 71 (1939), pp. 63–73) is madet to the solution of the following system which occurred in a problem of applied mechanics (I. Opatowski, Memorie, Accademia delle Scienze di Torino, vol. 69 (1938–1939), pp. 145–198): $d\eta_i(t, \xi)/dt = F_i[t, \xi, \xi'_0; \eta(t, \xi); \eta(t, \xi'_0)]$, where $i=1, \cdots, n$; ξ, ξ' and η mean $\xi_1, \cdots, \xi_m$; $\xi'_1, \cdots, \xi'_m$; $\eta_1, \cdots, \eta_n$, respectively; $\eta_i(t, \xi)$ are unknown functions with the initial conditions $\eta_i(0, \xi)=\bar{\eta}_i(\xi)$ and F_i are functionals eliminating ξ'. (Received July 31, 1939.)

358. G. H. Peebles: *On equivalence of certain types of series of orthonormal functions.*

This paper extends a theorem of equivalence for orthonormal polynomials in a single variable (see G. H. Peebles, Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 97–104) to somewhat more general systems of orthonormal functions, including, for example, polynomials in two real variables orthonormal on a suitable algebraic curve. (Received July 31, 1939.)

359. I. E. Perlin: *Indefinitely differentiable functions with prescribed least upper bounds.*

Let $F(x)$ be a real-valued indefinitely differentiable function of the real variable x defined on the interval $0 \leqq x \leqq a$. Let M_i denote the least upper bound of $|F^{(i)}(x)|$ on this interval. The author establishes sufficient conditions in order that there exist an indefinitely differentiable function with certain prescribed M_i. (Received July 24, 1939.)

360. B. J. Pettis: *Absolutely continuous functions in vector spaces.*

In an arbitrary space T let ξ be a Borel family of subsets that includes T and let $\alpha(E)$ be a nonnegative function completely additive over ξ, with $T=\sum T_i$ where $\alpha(T_i)<\infty$. Suppose $x(E)$, defined from ξ to a Banach space X, satisfies this condition: for each f in X^*, the adjoint of X, the numerical function $f(x(E))$ is completely additive and vanishes whenever $\alpha(E)=0$. Then $x(E)$ is completely additive, and given $\epsilon>0$ there is a $\delta>0$ such that $\|x(E)\|<\epsilon$ whenever $\alpha(E)<\delta$. From this and the Vitali-Hahn-Saks theorem flow easily the chief results connected with unconditional (or complete) convergence in Banach spaces. Applications to abstract integration and linear operations also can be made. For example, let L be the space of real functions integrable over T with respect to ξ and α; then if X^* is weakly complete, every linear operation from X to L is weakly completely continuous. (Received July 31, 1939.)

361. B. J. Pettis: *The uniform differentiation of families of numerical functions.*

The paper consists of applications of results obtained previously (Duke Mathematical Journal, vol. 5 (1939), pp. 254–269). A typical theorem is the following: On $T=(0, 1)$ let $\alpha_n(t)$, $n=1, 2, \cdots$, be a sequence of numerical functions converging everywhere and satisfying the condition $\sup_\pi \sum_{i=1}^{m} \sup_n |\alpha_n(t_i)-\alpha_n(t_{i-1})| < \infty$, π varying over the partitions $\pi=[t_0=0<t_1<t_2<\cdots<t_m=1]$ of $(0, 1)$. If the derivatives $\alpha_n'(t)$, which exist almost everywhere, also converge almost everywhere, then there is a set T_0 of measure zero such that $t \in T-T_0$ implies that $\alpha_n(t)$ is differentiable to $\alpha_n'(t)$ at t_0 uniformly with respect to n. (Received July 31, 1939.)

362. Everett Pitcher: *Critical points of a map to a circle.*

Let M be an n-dimensional manifold and F a function on M to the oriented circle. The manifold and the map are assumed to have suitable differentiability in local coordinate systems. Singular chain topology with integers modulo 2 as coefficients is used. The paper considers differential critical points and assumes that their images are isolated. A set of type numbers is assigned to each connected critical set as in the

local definition in the theory of critical points of a real valued function. Let M_k denote the sum of the kth type numbers over all connected critical sets. The following theorem is proved: There are numbers Q_k, positive or zero, which are invariants of the homotopy class of F and which together with the numbers M_k satisfy the relations $M_k - M_{k-1} + \cdots + (-1)^k M_0 \geqq Q_k - Q_{k-1} + \cdots + (-1)^k Q_0$. Here k ranges from 0 to n and the equality holds when $k = n$. This result is sharper than that in the author's abstract 45-1-53. Two equivalent definitions of the numbers Q_k are given. The present method depends on the use of an *induced covering space.* (Received July 31, 1939.)

363. Everett Pitcher: *Identification of two subsets.*

Let K denote an n-dimensional complex and C_1 and C_2 separated subcomplexes which are copies of a complex C. Let L denote the complex obtained from K by identifying cells of C_1 and C_2 which are copies of the same cell of C. Let B^k denote a k-dimensional homology group (coefficients from an arbitrary group). Let N^k denote the group of *sections*, that is, homology classes of k-cycles on C for which the difference of the copies on C_1 and C_2 bounds on K. Let S^k denote the group of homology classes of k-cycles on L which are images of k-cycles on K in the map on K to L. Let T^k denote the group of homology classes on K which contain the differences of the pairs of copies of k-cycles of C. The paper proves the following theorem: I. $B^k(L) \bmod S^k(L) \approx N^{k-1}$; II. $B^k(K) \bmod T^k \approx S^k$; III. $B^k(C) \bmod N^k \approx T^k$; $k = 0, 1, \cdots, n$. Weaker results in this direction were obtained by A. B. Brown. A variety of extensions are obtained, including to the case in which C_1 and C_2 are not separated, to singular chain and Vietoris topologies, and to cocycles. (Received July 31, 1939.)

364. W. T. Puckett: *On regular transformations.*

The continuous transformation $T(M) = M'$ is said to be $(r-1)$-regular (see A. D. Wallace, abstract 44-3-16) if for every sequence of points $\{a_n'\}$ converging to a' in M' the sequence $\{T^{-1}(a_n')\}$ converges s-regularly (see Whyburn, Fundamenta Mathematicae, vol. 25, pp. 408–426) to $T^{-1}(a')$ for every $s \leqq r-1$. If M is a continuum and a', b' are any two points of M', then the s-dimensional Betti groups of $T^{-1}(a')$ and $T^{-1}(b')$ relative to homologies in M are isomorphic for $s = 0, 1, \cdots, r$. Furthermore, in case T is a monotone 0-regular transformation, the 1-dimensional Betti group of M is the direct sum of two groups, one of which is isomorphic with the 1-dimensional Betti group of M', while the other is isomorphic with the 1-dimensional Betti group of $T^{-1}(a')$ relative to M for any point a' of M'. Thus $p^1(M) = p^1(M') + p^1(T^{-1}(a'), M)$, where $p^1(N)$ is the first Betti number of N and, for any point a' of M', $p^1(T^{-1}(a'), M)$ is the number of linearly independent 1-dimensional convergent cycles in $T^{-1}(a')$ relative to homologies in M. (Received July 10, 1939.)

365. W. T. Reid: *A theorem on continuous functions in abstract spaces.*

This note is concerned with a theorem on continuous functions in a very general abstract space, by means of which there are deduced certain results concerning semimetric spaces. In particular, the author generalizes a theorem of Montgomery (this Bulletin, vol. 40 (1934), pp. 620–624) concerning the behavior of distances between points of a metric space under transformations of the space into itself. (Received July 31, 1939.)

366. W. T. Reid: *Sufficient conditions by expansion methods for multiple integral problems of the calculus of variations.*

The present paper is concerned with sufficient conditions for a strong relative minimum for multiple integral problems of the calculus of variations, the method of proof being a refinement of the expansion method introduced for a simple integral problem by E. E. Levi. The results obtained extend those of Miranda (Memorie, Reale Accademia d'Italia, vol. 5 (1934), pp. 159–172) in not only the consideration of more general multiple integral problems, but also in the use of the Haar form of the Euler and Jacobi equations. The refinement of proof corresponds to that already given by the author in extending Levi's method to obtain sufficient conditions for the problem of Bolza, both in non-parametric and parametric form (Annals of Mathematics, (2), vol. 38 (1937), pp. 662–678; Transactions of this Society, vol. 42 (1937), pp. 183–190; also, abstract 42-9-361). (Received July 31, 1939.)

367. J. F. Ritt and H. W. Raudenbush: *Ideal theory and algebraic difference equations.*

There is derived a basis theorem for infinite systems of difference polynomials. A restricted theory of ideals is then obtained. (Received July 5, 1939.)

368. M. S. Robertson: *Typically-real functions with $a_n=0$ for $n\equiv 0 \bmod 4$.*

Within the unit circle the analytic function $f(z)=z+\sum_2^\infty a_n z^n$, $a_n=0$ for $n\equiv 0 \bmod 4$, takes on real values when and only when z is real. For functions of this type the author obtains the following new and sharp inequalities: For m and n both odd and $n>1$, $|a_n|+2^{-3/2}[(n-2)|a_{2m}|+n|a_2|]\leqq n$, $|a_n|+2^{-1/2}(n-1)|a_2|\leqq n$. For n even, $|a_n|+|a_2|\leqq 2^{3/2}$, $|a_2|\leqq 2^{1/2}$, $\limsup |a_n/n|\leqq 1-2^{-1/2}|a_2|$. The equality signs hold in each case for the univalent function $z(1-2^{1/2}z+z^2)^{-1}=2^{1/2}\sum_{n=1}^\infty z^n \sin n\pi/4$. (Received July 31, 1939.)

369. A. E. Ross: *A note on representation of integers by positive quaternary quadratic forms.*

In this paper the representation of "large" (see Ramanujan, Proceedings of the Cambridge Philosophical Society, vol. 19 (1916), pp. 11–21; Kloosterman, Acta Mathematica, vol. 49 (1926), pp. 407–464; also cf. Dickson, *Modern Elementary Theory of Numbers*, The University of Chicago Press, 1939) integers by positive (classic) quaternary quadratic forms is studied. A canonical form with suitably chosen leading principal minors is derived and with the help of this canonical form conditions are determined under which "large" integers are represented. (Received August 1, 1939.)

370. W. E. Roth: *On the unilateral equation in matrices.* II.

Given the equation $\sum_{i=0}^{p} A_i X^i=0$, where the A_i are $m\times n$ matrices with elements in F. Solutions of this equation whose characteristic determinants are not entirely reducible to linear factors in F are obtained by operations in F. The results parallel those of an earlier paper by the author (Transactions of this Society, vol. 32 (1930), pp. 61–68). (Received August 4, 1939.)

371. A. C. Schaeffer and R. J. Duffin: *A refinement of the inequality of the brothers Markoff.*

Let $f(x)$ be a polynomial of degree n or less which satisfies the inequality $|f(x)| \leq 1$ in the interval $-1 \leq x \leq 1$. The theorem of A. Markoff states that under these conditions $|f'(x)| \leq n^2, (-1 \leq x \leq 1)$. The theorem of W. Markoff gives the least upper bounds for the higher derivatives of $f(x)$. This paper observes that the stated conditions for these theorems are unnecessarily restrictive. Instead of assuming that $|f(x)| \leq 1$ at all points of the interval it is sufficient to assume that this inequality is satisfied only at the $n+1$ points $x = \cos k\pi/n$, $(k = 0, 1, \cdots, n)$; and the conclusions remain unchanged. (Received August 3, 1939.)

372. Edith R. Schneckenburger: *On 1-bounding monotonic transformations which are equivalent to homeomorphisms.*

A 1-bounding transformation of a space A onto a space B is a transformation such that for each point b of B the one-dimensional Betti number of the inverse of b is zero. A transformation $T(A) = B$ is equivalent to a homeomorphism when B is homeomorphic to A. Let T represent a single-valued, continuous, 1-bounding, monotonic transformation. This paper gives: (1) a necessary and sufficient condition that T be equivalent to a homeomorphism when A is a finite graph; (2) a sufficient condition that T be equivalent to a homeomorphism when A is a tree. It is shown also that when A is a Peanian continuum imbedded in a two-manifold M and every 1-bounding monotonic T is equivalent to a homeomorphism: (a) if A is cyclic, A is a simple closed curve or the two-manifold M; (b) if A is acyclic, A is an arc; (c) if A is neither cyclic nor acyclic and certain restrictions are put upon the cut points of A, then A is a finite or countably infinite set of simple closed curves tangent to each other at a single point. (Received July 17, 1939.)

373. I. J. Schoenberg: *On local convexity in euclidean spaces.*

H. Tietze (Mathematische Zeitschrift, vol. 28 (1928), pp. 697–707) introduced the following notion of local convexity: Let M be a set in the euclidean space E_k. Let $P \in M$. The set M is said to be *locally convex at* P, if there is a positive $\rho = \rho(P)$ such that the intersection $M \cdot S(P; \rho)$ of M with the open sphere $S(P; \rho)$, of center P and radius ρ, is convex in the ordinary sense. The set M is said to be *locally convex* if it is locally convex at all its points. Tietze's chief result is the following theorem: A continuum M which is locally convex is convex in the ordinary sense. A short and simple proof of Tietze's theorem is given in this note. It operates with the notion of uniform local convexity. (Received July 31, 1939.)

374. Abraham Schwartz: *The Gauss-Codazzi-Ricci equations in Riemannian manifolds.*

The Gauss-Codazzi-Ricci equations for a $V_m \subset V_n$ are considered in the form in which they are given by J. A. Schouten and D. J. Struik (*Einführung in die neueren Methoden der Differentialgeometrie*, Groningen, 1938, vol. 2, pp. 121 ff.). It is shown that each of these equations is a consequence of others under simple conditions on the curvature affinors of valence 3. These conditions can be expressed geometrically in terms of conjugate vectors on the V_m or their analogues in the various normal spaces. Algebraically, when the normal space considered is of 1 or 2 dimensions, they can be expressed in terms of the invariants describing certain matrices, which matrices reduce to the second fundamental form in a particular case, $V_{n-1} \subset V_n$. As an illustra-

tion, the Gauss equation referring to the $(x+1)$st normal space is a consequence of the Gauss, Codazzi, and Ricci equations referring to the xth normal space and of the Ricci equation referring to the $(x+1)$st normal space if the curvature affinor referring to the xth normal space satisfies a certain condition. The result of T. Y. Thomas (*Riemann spaces of class one and their characterization*, Acta Mathematica, vol. 67 (1935), pp. 189 ff.) is included in a particular case of a theorem here. (Received July 27, 1939.)

375. A. R. Schweitzer: *On a classification of "configurational" sets of ordered dyads.*

In continuation of a previous paper reported in this Bulletin (abstract 44-11-474) configurational sets of ordered dyads are classified into types, first with reference to "connection" in the sense of the author's theory (American Journal of Mathematics, vol. 31, chap. 2) and second with reference to "right" (R) and "left" (L). The first classification distinguishes between completely connected, partially connected and completely disconnected sets. The second classification distinguishes between "R" and "L" sets in the author's genetic construction of complete systems of configurational sets by making the following conventions: 1. A set consisting of a single self-conjugate dyad is an R set. 2. An $R(L)$ set remains an $R(L)$ set after an adjunction. 3. An $R(L)$ set becomes an $L(R)$ set after a replacement. Application is made to groups of configurational sets and also to the construction of the sequence of abstract determinants: $R(\alpha\alpha)$, $R(\alpha\alpha, \beta\beta)+L(\alpha\beta, \beta\alpha)$, and so on. Reference is made to Chapter 3 of the author's theory (ibid., vol. 31), to Cayley (Journal für die reine und angewandte Mathematik, vol. 38, p. 93), and Grassmann (*Gesammelte Werke*, vols. 1–2, §504). (Received July 29, 1939.)

376. A. R. Schweitzer: *On a classification of systems of "configurational" sets of ordered dyads.*

The author classifies as follows. A. Complete systems. Represent groups. B. Incomplete systems. I. Represent groups and include systems which permit definition of finite (Galois) fields by introducing a relation of equality among the dyads of each configurational set. II_1. Do not represent groups but allow interpretation as finite geometries including pseudodescriptive geometries (American Journal of Mathematics, vol. 31, p. 379). II_2. Do not represent groups but permit definition of configurations (1) with, (2) without, reference to order and include systems which allow definition of finite nets in projective geometry. C. Systems which represent some or all of the preceding classes in relation to one another. Reference is made to the author's foundations of projective geometry (this Bulletin, abstract 42-11-443), the tactical configurations of E. H. Moore and others, and the finite geometries of Veblen and others. (Received July 29, 1939.)

377. H. A. Simmons and J. F. Paydon: *On the shortest time of fall from a curve to a point.*

A well known problem related to the present note is to determine on a curve Γ of class C', in a vertical plane, the point 1 from which a particle will fall from rest under gravity to a fixed point 2 in the shortest time. The purpose of this note is to associate with Γ, when it is mildly restricted, an envelope G such that the tangent through 2 to G intersects Γ in the point 1. The envelope G plays here a role analogous to that which is played by the envelope, when existent, of the family of normals to

an arc in the problem of *finding the shortest distance from a point to an arc.* (Received July 31, 1939.)

378. Andrew Sobczyk: *Projections in Minkowski and Banach spaces.*

The results of Murray (Transactions of this Society, vol. 41 (1937), p. 138) on projections in l_p spaces, and additional quantitative information, are obtained by a briefer method. Various generalizations of l_p spaces are studied, in particular, spaces defined by a sequence $p_1, p_2, p_3, \cdots$ of exponents. For these spaces it is shown that if a projection exists on every closed linear subspace, the p_i's must converge to 2 so rapidly that the space is isomorphic to Hilbert space. A similar result is obtained for the spaces of Orlicz. (Received July 28, 1939.)

379. N. E. Steenrod: *Homology groups of the complement of a closed set in an n-sphere.*

Let S^n be an n-sphere and C a closed subset and let $S^n - C$ be subdivided into an infinite complex K. Infinite q-chains in K with coefficients in a topological abelian group G are considered. Let Z_q be the group of q-cycles, B_q the group of bounding cycles and $\overline{B}_q$ its closure (if G is compact or if C is a neighborhood retract, $B_q = \overline{B}_q$; but this is not true in general otherwise). It is well known that $\overline{H}_q = Z_q - \overline{B}_q$ is isomorphic with the Vietoris homology group of C of dimension $q-1$; it is therefore a topological invariant of C. In this paper $H_q = Z_q - B_q$ is proved to be a topological invariant of C. A method is given for defining H_q for an arbitrary compact metric C directly in terms of the topology of C. A study is made of H_1 where C is a solenoid and G is the discrete group of integers. It is proved that H_1 has the power of the continuum. This is in contrast to $\overline{H}_1 = 0$. It follows from results of Eilenberg that there is a continuum of distinct homotopy classes of maps of $S^3 - C$ into S^2. This answers a question raised by Borsuk and Eilenberg. (Received July 31, 1939.)

380. Wolfgang Sternberg: *The general limit theorem in the theory of probability.*

Let x_i, $(i = 1, 2, \cdots, n)$, be independent stochastic variables with any distribution functions V_i. Then the general limit theorem states: The distribution function $W_n(x)$ of the sum $x_1 + x_2 + \cdots + x_n$ converges (as n increases indefinitely) uniformly in x to the Gaussian function ϕ, when very general assumptions with respect to the V_i are made. Denoting standard deviation, an absolute moment of the third order of x_i, by b_i and c_i, respectively, and putting $b_1 + \cdots + b_n = B_n$ the following assumptions are made: $\lim_{n\to\infty} b_i/B_n = V$, $\lim_{n\to\infty} c_i/b_i(B_n)^{1/2} = V$ uniformly for all $i = 1, 2, \cdots, n$. A proof of the theorem is given, using the recursion formula $W_i(x) = \int_{-\infty}^{+\infty} W_{i-1}(x-\xi)\,dV_i(\xi)$ and the fact that the Gaussian function $\phi(x/(2y)^{1/2})$ satisfies the equation $\partial\phi/\partial y = \frac{1}{2}\partial^2\phi/\partial x^2$. One should compare this with the proof published by A. Khintchine (*Asymptotische Gesetze der Wahrscheinlichkeitsrechnung*, Berlin, 1933), which is based on works of F. Petrowsky and A. Kolmogoroff. Khintchine's proof gave the suggestion for the present paper. Other assumptions have been chosen in order to make the exposition as simple as possible, while Khintchine strives for wide generality and so assumes nothing as to moments of the third order. A so-called upper- and lower-function, which Khintchine introduces as the fundamental idea of the Petrowsky-Kolmogoroffian proof, is not used by the author. (Received July 25, 1939.)

381. R. M. Thrall: *A note on groups of class 3.*

In this note, formulas are given for commutator computation in all groups of class 3. These formulas are applied to investigate the properties of groups of class 3 with all operators not equal to 1 of prime order $p>3$. If such a group has n independent generators, it is a quotient group of a particular group A_n of order p^N, $N=C_{n,1}+3C_{n,2}+2C_{n,3}$ with n independent generators. The only characteristic subgroups of A_n are the commutator subgroup and the central. The group A_n is a generalization for $p>3$ of the group A_n for $p=3$ investigated by Levi and van der Waerden (Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 9 (1933), pp. 154–158). (Received July 31, 1939.)

382. W. J. Trjitzinsky: *General theory of singular integral equations with real kernels.*

The theories of integral equations of importance from the present point of view are those due to Volterra, Fredholm, Hilbert, E. Schmidt, Carleman. The developments of Carleman present the culminating aspects in the above sequence; moreover, his theory has already had some particularly important applications (Schrödinger wave equation, nonlinear differential equations of the type occurring in dynamics). In the present work is given an extensive and systematic treatment of equations with kernels of any finite or transfinite rank. Carleman's kernels are designated as of rank one. Kernels of rank $\alpha(>1)$ are limits (in one sense or another) of kernels of lower ranks. Thus there arises a transfinite hierarchy of theories of ascending generality. (Received July 31, 1939.)

383. C. W. Vickery: *A theorem on abstract sets.*

Suppose that S is a set, $\aleph_\alpha$ a cardinal number, and π a property such that (1) every subset of S that has property π is of power greater than or equal to $\aleph_\alpha$, (2) the collection of all subsets of S having property π is of power $\aleph_\alpha$, then for every ordinal $\beta<\alpha$ there exists a collection G_β of power $\aleph_\beta$ of mutually exclusive subsets of S, each of power $\aleph_\alpha$, no one of which contains a subset having property π and such that every subset of S that has property π contains an element of each set G_β. The proof involves Zermelo's well ordering theorem. (Received July 19, 1939.)

384. C. W. Vickery: *On spaces (V_{ω_α}) and spaces Λ_{ω_α}.*

A space (V) of Fréchet is said to be a space (V_{ω_α}) provided every point is of type ω_α. In order that a space (V_{ω_α}) be a space Λ_{ω_α} (C. W. Vickery, Tôhoku Mathematical Journal, vol. 40 (1934), p. 6) it is necessary and sufficient that, for each two distinct points p and q, there exist neighborhoods R_p of p and R_q of q such that $R_p \cdot R_q=0$. A generalization of condition I of Fréchet's *Les Espaces Abstraits*, p. 217, is a necessary and sufficient condition that a space Λ_{ω_α} be a space (V_{ω_α}). (Received July 19, 1939.)

385. C. W. Vickery: *On two different types of linear continua.*

Examples of several types of linear continua have been given by the author (Tôhoku Mathematical Journal, vol. 40 (1934), pp. 1–26). Space U_{1,ω_1} represents the greatest extension of the real number continuum which preserves its local properties. Consider the following axioms: (1) S is an ordered set (with respect to $\prec$) of at least two points. (2) If $a\prec b$, there exists c such that $a\prec c\prec b$. (3) Dedekind's postulate is satisfied. (4) There exists no first or last point. (5) Every point is the limit of a monotonic ascending and a monotonic descending type ω sequence of points. (6) S is com-

pact. (6′) There exist two type ω monotonic sequences of points, one ascending and the other descending, neither of which has a limit point. Axioms 1–6 are independent and characterize space U_{1,ω_1}. Axioms 1–6′ are independent and characterize the real number continuum. Axioms 1–5 and the denial of 6 necessitate that S be either a ray of U_{1,ω_1} or the real number continuum. (Received July 19, 1939.)

386. C. W. Vickery: *Types of limit points in spaces (V).*

Suppose S is a space (V) of Fréchet. A point p of S is said to be of *order* $\aleph_\alpha$ provided there exists a collection V_p of power $\aleph_\alpha$ such that if M is a point set, p is a limit point of M if and only if every neighborhood of V_p contains a point of M distinct from p and provided $\aleph_\alpha$ is the smallest cardinal for which this is true. The space is said to be *uniform* at p provided there exists a monotonic descending sequence γ_p of neighborhoods of p such that (1) no point other than p is common to all elements of γ_p and (2) if R_p is a neighborhood of p, there exists an element of γ_p which is a subset of R_p; p is said to be of *type* ω_α provided ω_α is the smallest ordinal of such a sequence γ_p. If p is a point of order $\aleph_\alpha$, in order that p be of type ω_α it is necessary and sufficient that, if G_p is a collection of neighborhoods of p of power less than $\aleph_\alpha$, there exist a neighborhood of p which is a subset of every element of G_p. (Received July 19, 1939.)

387. A. D. Wallace: *On quasi-monotone transformations.*

The transformation $T(A)=B$ is said to be quasi-monotone provided that for each open connected set R of B the set $T^{-1}(R)$ has only a finite number of components each of which maps onto all of R. If A is a locally connected continuum then T is quasi-monotone if and only if it is the product of a monotone and a light interior transformation. It follows that a topological property is invariant under quasi-monotone transformations if and only if it is invariant under both monotone and light interior transformations. Certain results of a previous communication (abstract 45-1-69) are extended from monotone to quasi-monotone (and hence to interior) transformations. (Received July 28, 1939.)

388. M. S. Webster: *Note on certain Lagrange interpolation polynomials.*

The fundamental Lagrange interpolation polynomial $l_k^{(n)}(x)\equiv\phi_n(x)/[(x-x_k)\phi_n'(x_k)]$, where $\phi_n(x)\equiv(x-x_1)(x-x_2)\cdots(x-x_n)$ and $-1<x_k<1$, is considered when $x_1, x_2, \cdots, x_n$ are the zeros of the Jacobi polynomial $\phi_n(x;\alpha,\beta)$ of nth degree which satisfies the differential equation $(1-x^2)\phi_n''(x)+[\alpha-\beta-(\alpha+\beta)x]\phi_n'(x)+n(n+\alpha+\beta-1)\phi_n(x)=0$. For special values of the parameters α, β, the least upper bound for $|l_k^{(n)}(x)|$ in $(-1, 1)$ is obtained. For example, if $\alpha=\beta=3/2$, the least upper bound is 2. Certain limiting relations are also given. (Received June 24, 1939.)

389. G. W. Whaples: *On the structure of moduls with a commutative algebra as operator domain.* Preliminary report.

The solutions T of the matrix equations $TA_i=B_iT$, $(i=1, \cdots, n)$, over field F, correspond to certain homomorphisms of the total vector space with left operators $F[I, A_1, \cdots, A_n]$ to subspaces of the total vector space with left operators $F[I, B_1, \cdots, B_n]$. A new method for determining the solutions of $TA_i=B_iT$, some theorems on the possible ranks of these solutions, and a representation for the ring of all matrices commutative with $A_1, \cdots, A_n$ are given for the case in which $F[I, A_1, \cdots, A_n]$ and $F[I, B_1, \cdots, B_n]$ are commutative. If R is a commutative

algebra with unit over an infinite separable field F, the minimum n such that $R=F[r_1, \cdots, r_n]$ is found; R is a principal ideal ring if and only if $n=1$. Many of the theorems on R-modules which are true for $n=1$ cannot be extended to the case $n>1$; for example, if $n>1$, R-modules are not always expressible as direct sums of cyclic R-modules. (Received August 3, 1939.)

390. P. M. Whitman: *Free lattices.*

It is proved that in the free lattice generated by a set of elements X_i, (1) $X_i \leqq X_j$ if and only if $i=j$, and (2) recursively, $A \leqq B$ if and only if one or more of the following are true: (a) $A=A_1 \cup A_2$, where $A_1 \leqq B$ and $A_2 \leqq B$, (b) $A=A_1 \cap A_2$, where $A_1 \leqq B$ or $A_2 \leqq B$, (c), (d) the dual conditions. A canonical form for the elements of such a lattice is deduced, and it is shown that if the number of generators is finite then certain elements cover others. (Received June 26, 1939.)

391. G. T. Whyburn: *A relation between interior and non-alternating transformations.*

Theorems are first established from which it follows that an interior transformation of a compact continuum into a dendrite will be non-alternating if and only if the inverse of each end point of the dendrite is connected. It is then shown that if M is a compact continuum and H is the set of all end points of a dendrite D, then any interior transformation $f(M)=D$ such that the collection $[f^{-1}(p)]$, $p \in H$, is semi-closed can be factored into the form $f=f_2 f_1$ where f_1 is equivalent to f on $f^{-1}(H)$ and topological on $M-f^{-1}(H)$ and f_2 is interior and non-alternating. (Received June 30, 1939.)

392. G. T. Whyburn: *On the interiority of real functions.*

In this paper it is shown that if $f(x)$ is a real-valued continuous function defined on a connected and locally connected separable metric space M (therefore transforming M into a set Y of real numbers), there exists a countable subset C of Y such that f is interior at every point of $M-f^{-1}(C)$. Also the following extension theorem is proven: If M is a subcontinuum of a cyclic locally connected compact continuum L and $f(M)=(0, 1)$ is continuous and such that for any y, $0 \leqq y \leqq 1$, $L-f^{-1}(y)$ is connected, there exists an extension $\phi(L)=(0, 1)$ of f to L which is interior at every point of $L-M$ and at every point of M where f is interior. (Thus if f is interior on M, ϕ is interior on L.) (Received June 30, 1939.)

393. L. R. Wilcox: *A new foundation for projective differential geometry.*

It was recently proved by von Neumann that the n-dimensional projective space L_n over a field F is isomorphic to the lattice of all (principal) right ideals in the ring F_n of all nth order matrices with elements in F. This paper assumes F to be real or complex and lays the foundations for a projective differential geometry of L_n wholly in terms of the algebraic and topological structure of F_n, the topology of F_n being that obtained by defining for $A \in F_n$ a norm $\|F\| \equiv (\text{trace } (A^*A))^{1/2}$. It is shown that this theory leads to classical projective differential geometry; differential equations of a configuration are found to appear as conditions on the trace of certain functions of the matrix representing the configuration. (Received August 1, 1939.)

394. Margarete C. Wolf: *Transformations of bases for relative sets over a non-commutative field.*

This is a continuation of a paper by the author (this Bulletin, vol. 44 (1938), pp. 716–718) for the case that R is a vector space of dimension n over a non-commutative field $\mathfrak{F}$, and M a matrix of a linear operator relative to a base of R. If $\xi_1, \xi_2, \cdots, \xi_k$ is a set of vectors such that R is a direct sum of the cyclic subspaces $L_M(\xi_i)$, $(i=1, 2, \cdots, k)$, the problem of finding conditions under which R can be expressed as a direct sum of other cyclic subspaces has been made to depend upon the unique solution of k sets of n linear equations in n unknowns with coefficients in $\mathfrak{F}$. In special cases several more specific results are obtained. For example, if $L_M(\xi_1) \cap L_M(\xi_2) = 0$ and if ξ_1 and ξ_2 are minimally associated relative to M with polynomials g_1 and g_2 respectively, then a necessary and sufficient condition that there exist a vector η such that $L_M(\eta) = L_M(\xi_1) \oplus L_M(\xi_2)$ is that g_1 and g_2 be similar to a pair of relatively prime polynomials. (Received July 25, 1939.)

395. J. W. T. Youngs: *Arc-spaces.*

Certain axioms in terms of *point* and *arc* determine arc-spaces. Cyclic elements are defined and it is noticed that the hyperspace of cyclic elements is itself an arc-space which is a dendrite. The development is analogous to that of cyclic element theory for Peano spaces, and one sense in which the hyperspace of cyclic elements of a Peano space may be said to be a dendrite is that it is an arcspace with one and only one arc joining any two of its points. (Received July 24, 1939.)

396. J. W. T. Youngs: *Generalized cyclic elements.*

If $C(ab)$ is the cyclic chain from a to b, then $C(ab)$ is a true cyclic element if and only if a is conjugate to b. The generalization to true k-cyclic elements is made from this point of view. A point a is said to be k-conjugate to b if no set of k distinct points separates a from b in the space. Roughly speaking, if $a_1, \cdots, a_{k+1}$ are distinct, and k-conjugate to each other then the totality of points k-conjugate to a_j for $=1, \cdots, k+1$, constitute a true k-cyclic element. In spite of the fact that a k-cyclic element is not, in general, connected, a large number of the desirable properties of cyclic elements may be extended to k-cyclic elements. (Received July 24, 1939.)

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

AGOSTINI, A. Trigonometria Piana e Sferica. Livorono, R. Accademia Navale, 1938. 218 pp.

BACHELIER, L. Les Nouvelles Méthodes du Calcul des Probabilités. Paris, Gauthier-Villars, 1939. 8+71 pp.

BELL, E. T. Les Grands Mathématiciens. Paris, Payot, 1939. 7+615 pp.

BON, F. Ist es wahr dass $2\times2=4$ ist? Vol. 2: Von den Kriterien der Wahrheit. 83 pp. Vol. 3: Von den mathematischen Grundbegriffen. 80 pp. Leipzig, 1939.

COLERUS, E. Piccola Storia della Matematica, da Pitagora a Hilbert. Turin, Einaudi, 1939. 359 pp.

DAUGHERTY, R. D. See STRATTON, W. T.

EISENHART, L. P. Coordinate Geometry. Boston, Ginn, 1939. 11+298 pp.

ENRIQUES, F. Le Matematiche nella Storia e nella Cultura: Lezioni Pubblicate per Cura di Attilio Frajese. Bologna, Zanichelli, 1938. 339 pp.

FAWCETT, H. P. The Nature of Proof. (National Council of Teachers of Mathematics Yearbook.) New York, Columbia University Press, 1938. 11+146 pp.

FINIKOFF, M. S. Déformation à Reseau Conjugué Persistant et Problèmes Géométriques qui s'y Rattachent. (Mémorial des Sciences Mathématiques, vol. 96.) Paris, Gauthier-Villars, 1939. 86 pp.

GREEN, S. L. The Theory and Use of the Complex Variable: An Introduction. London, Pitman, 1939. 8+136 pp.

HLAVATÝ, V. Differentialgeometrie der Kurven und Flächen und Tensorrechnung. Translated by M. Pinl. Groningen, Noordhoff, 1939. 11+569 pp.

KANTOROWITSCH, M. See TSCHUPROW, A. A.

KENNEY, J. F. Mathematics of Statistics. Part 1. 10+248 pp. Part 2. 9+202 pp. New York, Van Nostrand, 1939.

MALENGREAU, J. Notice sur les Fondements de la Géométrie. Brussels, De Boeck, 1939. 48 pp.

VON MISES, R. Probability, Statistics and Truth. Translated by J. Neyman, D. Sholl, and E. Rabinowitsch. London, Hodge; New York, Macmillan, 1939. 16+323 pp.

NEWMAN, M. H. A. Elements of the Topology of Plane Sets of Points. Cambridge, University Press, 1939. 8+221 pp.

NEYMAN, J. Lectures and Conferences on Mathematical Statistics. Washington, U. S. Department of Agriculture, 1938. 167 pp.

——— See VON MISES, R.

PINL, M. See HLAVATÝ, V.

RABINOWITSCH, E. See VON MISES, R.

SCHOLZ, A. Einführung in die Zahlentheorie. (Sammlung Göschen, no. 1131.) Berlin, de Gruyter, 1939. 136 pp.

SHIVELY, L. S. An Introduction to Modern Geometry. New York, Wiley; London, Chapman and Hall, 1939. 11+167 pp.

SHOLL, D. See VON MISES, R.

SIMONS, L. G. Fabre and Mathematics and Other Essays. (Scripta Mathematica Library, no. 4.) New York, Scripta Mathematica, 1939. 5+101 pp.

STRATTON, W. T., and DAUGHERTY, R. D. Plane Trigonometry. With Tables. New York, Prentice-Hall, 1939. 7+206 pp.

TSCHUPROW, A. A. Principles of the Mathematical Theory of Correlation. Translated by M. Kantorowitsch. London, Hodge, 1939. 10+194 pp.

VILLE, J. Étude Critique de la Notion de Collectif. (Monographies des Probabilités, no. 3.) Paris, Gauthier-Villars, 1939. 144 pp.

VAN DER WAERDEN, B. L. Einführung in die algebraische Geometrie. (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 51.) Berlin, Springer, 1939. 7+247 pp.

PART II. APPLIED MATHEMATICS

COHN, B. Tables of Addition and Subtraction Logarithms with Six Decimals. 2d edition. Preface by L. J. Comrie. London, Scientific Computing Service, 1939. 8+63 pp.

COMRIE, L. J. See COHN, B.

DÖRGE, K., and KLEIN, H. Wahrscheinlichkeitsrechnung für Nichtmathematiker. Berlin, de Gruyter, 1939. 113 pp.

FREEMAN, H. Mathematics for Actuarial Students. Part 1: Elementary Differential and Integral Calculus. 8+184 pp. Part 2: Finite Differences, Probability and Elementary Statistics. 13+340 pp. Cambridge, University Press, 1939.

KLEIN, H. See DÖRGE, K.

MCLACHLAN, N. W. Complex Variable and Operational Calculus with Technical Applications. Cambridge, University Press, 1939. 11+355 pp.

REED, C. E. See SHERWOOD, T. K.

ROWE, C. E. Engineering Descriptive Geometry. New York, Van Nostrand, 1939. 8+299 pp.

SCHLAUCH, W. S. Business Arithmetic for College Students. New York, Crofts, 1939. 7+299 pp.

SHERWOOD, T. K., and REED, C. E. Applied Mathematics in Chemical Engineering. (Chemical Engineering Series.) New York and London, McGraw-Hill, 1939. 11+404 pp.

SOLOMON, J. Protons, Neutrons, Neutrinos. (Collection de Physique Mathématique, no. 6.) Paris, Gauthier-Villars, 1939. 13+228 pp.

TINBERGEN, J. Les Fondements Mathématiques de la Stabilisation du Mouvements des Affaires. (Actualités Scientifiques et Industrielles, no. 632.) Paris, Hermann, 1938. 114 pp.